Calculus (Type A. 3 Pages)

Part I: Multiple-answer questions. For these questions, your answer should be one or more. To score all the points for each question, you must select ALL of the correct answers and NONE of the incorrect answers. Missing a correct answer or taking an incorrect one, you will lose 3 points. In the other cases, you will get zero.

1. (6 pts.) Which of the following statements are NOT correct?

   Ⓐ If $\lim_{x \to c} f(x) = 0$, then $\lim_{x \to c} |f(x)| = 0$;
   Ⓑ If $\lim_{x \to c} f(x) = L$, then $f(c) = L$;
   Ⓒ If $f(c) = L$, then $\lim_{x \to c} f(x) = L$;
   Ⓓ If $\lim_{x \to a} [f(x) + g(x)]$ exists, then $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ also exist.

   Ans: Ⓑ Ⓒ Ⓓ

2. (6 pts.) Let

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

   Which of the following statements are NOT correct?

   Ⓐ $\lim_{x \to 0} f(x) = \lim_{x \to 0} x \cdot \lim_{x \to 0} \sin(1/x) = 0 \cdot \lim_{x \to 0} \sin(1/x) = 0$;
   Ⓑ f is not continuous at $x = 0$;
   Ⓒ f is continuous on $(-\infty, \infty)$;
   Ⓓ f is differentiable on $(-\infty, \infty)$, and $f'(x) = \sin(1/x) + (x \cos(1/x))(-x^{-2})$.

   Ans: Ⓐ Ⓑ Ⓓ

3. (6 pts.) Which of the following statements are correct?

   Ⓐ Let $f(x)$ and $g(x)$ be continuous on $[a, b]$. If $f(a) < g(a)$ and $f(b) > g(b)$, then there exists $c \in (a, b)$ such that $f(c) = g(c)$;
   Ⓑ If $f(x) = \pi^2$, then $f'(x) = 2\pi$;
   Ⓒ If f is not continuous at $x = c$, then f is not differentiable at $x = c$;
   Ⓓ If f and g are differentiable, then $\dfrac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$.

   Ans: Ⓐ Ⓒ

4. (6 pts.) Which of the following statements are NOT correct for the real function $f(x)$ defined on (a, b)?

   Ⓐ If $f'(c) = 0$ for some point $c \in (a, b)$, then $f(x)$ attains a local extremum at $x = c$;

Ⓑ If $f(x)$ is continuous on (a, b), then $f(x)$ must have a global (absolute) minimum on (a, b);

Ⓒ If $f^2(x)$ is differentiable on (a, b), then $f(x)$ is also differentiable on (a, b);

Ⓓ If $f(x)$ is concave upward on (a, b), then $f(x)$ is increasing on (a, b).

Ans: Ⓐ Ⓑ Ⓒ Ⓓ

5. (6 pts.) Let

$$f(x) = \frac{x^2 + 2x - 8}{(x + 2)(x - 2)}.$$

Which of the following statements are correct?

Ⓐ $x = 2$, $x = -2$ are the vertical asymptotes of $f(x)$;

Ⓑ $\lim_{x \to -2^-} f(x) = \infty$;

Ⓒ $\lim_{x \to -2^+} f(x) = \infty$;

Ⓓ $y = 1$ is the horizontal asymptote.

Ans: Ⓒ Ⓓ

6. (6 pts.) Suppose that $f'(a) = 1$ where $a > 0$. The possible values for $\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - 1}$ are

Ⓐ 0;

Ⓑ 1;

Ⓒ 2;

Ⓓ 3.

Ans: Ⓐ Ⓒ

7. (6 pts.) Which of the following statements are NOT correct?

Ⓐ If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is a constant function on (a, b);

Ⓑ If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x)$ on (a, b);

Ⓒ If $f'(x)$ exists for all $x \in (-\infty, \infty)$ and $\lim_{x \to \infty} f(x) = c$ $(c \in \mathbb{R})$, then $\lim_{x \to \infty} f'(x) = 0$;

Ⓓ If $f'(x)$ and $g'(x)$ exist for all $x \in (a, b)$, then $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$ for some $\xi \in (a, b)$.

Ans: Ⓑ Ⓓ

Part II: Fill in the Blanks.

(a) (7 pts.) Find a and b such that $y = (a - 2)x + b$ is the tangent line to the graph of $f(x) = \frac{1}{x}$ at (-1,-1). $(a, b) =$ ____(⑧, ⑨⑩)____.

Ans: $(a, b) = (1, -2)$

(b) (7 pts.) If $\lim_{x \to -3} \frac{(f(x) - 4)(x + 3)}{x^2 + 4x + 3} = -1$, then $\lim_{x \to -3} f(x) =$ ____⑪____.

Ans: 6

(c) (7 pts.) Find $\lim_{x\to\infty} -x(\sqrt{x^2-x}-x)\sin\frac{1}{x} = \underline{\frac{⑫}{⑬}}$. (Reduce the answer to the simplest term.)

Ans:

$$\begin{aligned}\lim_{x\to\infty} -x(\sqrt{x^2-x}-x)\sin\frac{1}{x} &= \lim_{x\to\infty} \frac{-(\sqrt{x^2-x}-x)(\sqrt{x^2-x}+x)}{(\sqrt{x^2-x}+x)} \frac{\sin 1/x}{1/x} \\ &= \lim_{x\to\infty} \frac{x}{(\sqrt{x^2-x}+x)} \frac{\sin 1/x}{1/x} \\ &= 1/2 \\ &= 0.5.\end{aligned}$$

(d) (7 pts.) Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 1/2$, $g'(1) = 2$ and $f'(3) = 9$. Find $r'(1) =$ ⑭⑮ .

Ans: $r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$. $r'(1) = f'(3) \cdot g'(2) \cdot h'(1) = 18$.

(e) (7 pts.) Let $f'(x) \geq 6$ for all $x \in [1, 4]$. Assume $f(1) = 9$ and $f(4) = 27$. Evaluate $f(3) =$ ⑯⑰ .

Ans: 21.

(f) (7 pts.) Find a and b such that f is differentiable everywhere. $(a, b) =$ (⑱, ⑲)

$$f(x) = \begin{cases} -\cos(x), & \text{if } x < 0, \\ (a-1)x + b - 3, & \text{if } x \geq 0. \end{cases}$$

Ans: $(a, b) = (1, 2)$

(g) (7 pts.) Find $\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{4}+h\right)-1}{h} =$ ⑳ .

Ans:

$$\lim_{h\to 0} \frac{\tan(\pi/4+h)-1}{h} = \sec^2\left(\frac{\pi}{4}\right) = 2.$$

(h) (7 pts.) Use the graph of $f(x)$ shown in Figure 1 and linear approximation to estimate $f(1.95) =$ 1.㉑㉒㉓ .

Ans: $f(1.95) = f(2-0.05) = f(2) + f'(2)(-0.05) = 1.025$.

(i) (7 pts.) A plane flies horizontally at an altitude of 7 km and passes directly over a tracking telescope on the ground (see Figure 2). When the angle of elevation is $\pi/6$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane travelling at that time?

Answer: $\frac{㉔㉕}{㉖}\pi$ km/min. (Reduce the answer to the simplest term.)

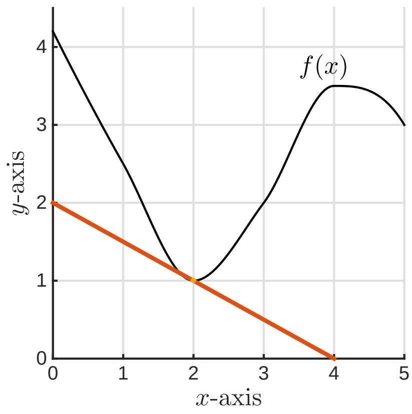

Figure 1

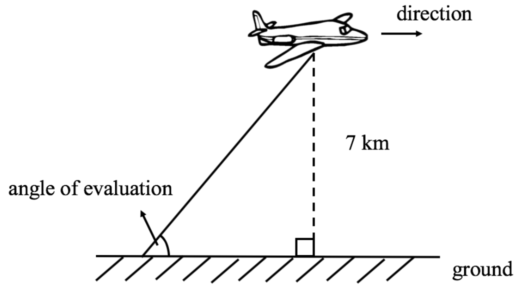

Figure 2

Ans: Let the angle of elevation be $\theta(t)$, and the horizontal displacement of the plane from the tracking telescope be $x(t)$, then from the figure we have $\tan\theta(t) = \frac{7}{x(t)}$. It can be derived that $x'(t)|_{\theta=\pi/3} = \frac{14}{3}\pi$ km/min.