Calculus Exam (Group A)

Part I: Multiple answer questions. To get all points for each question, you must select ALL correct choices and NONE of incorrect choices. If you miss a correct choice or taking an incorrect choice, then you will lose 50% of the full points. For all other cases you will get zero points.

Problem 1. (7 points) *A function $f : \mathbb{R} \to \mathbb{R}$ is called* ***one-to-one*** *if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in \mathbb{R}$. Let f, g be two one-to-one functions. Which of the following functions must be one-to-one?*

(A) *Their addition $f + g$.*

(B) *Their product fg.*

(C) *The quotient f/g.*

(D) *The composition $f \circ g$.*

Answer: (D)

Problem 2. (7 points) *Let f be a function satisfying $\lim_{x\to 0} xf(x) = 1$. Which of the following statements are correct?*

(A) $\lim_{x\to 0} f(x)$ *exists.*

(B) $f(0)$ *must be defined.*

(C) $\lim_{x\to 0} \sin(x)f(x) = 1$.

(D) $\lim_{x\to 0} x^2 f(x) = 0$.

Answer: (C), (D)

Problem 3. (8 points) *Let f be a function satisfying $f'(a) = 1$. Which of the following statements are correct?*

(A) $\lim_{h\to 0} \frac{f(a+2h)-f(a)}{h} = 2$

(B) $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{h} = 1$.

(C) $\lim_{h\to 0} \frac{f(a+h^2)-f(a)}{h} = 0$.

(D) $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h^2} = 0$.

Answer: (A), (C)

Problem 4. (8 points) *Consider the function* $f(x) = x^2|x|$. *Which of the following statements are correct?*

(A) $f'(0) = 0$.

(B) $f''(0) = 0$.

(C) $f'''(0) = 0$.

(D) $f''''(0) = 0$.

Answer: (A), (B)

Problem 5. (9 points) *Consider the function*

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Which of the following statements are correct?

(A) $f(x) > 0$ *for all* $x > 0$.

(B) $\lim_{x\to 0} f(x) = 0$.

(C) $f'(0) = 0$.

(D) $\lim_{x\to 0} f'(x) = 0$.

Answer: (B), (C)
From the textbook §**2.1** P. 58.

Problem 6. (9 points) *Let* $y = y(x)$ *be the implicit function defined by the equation* $x^2+y^3 = xy+3$ *near* $(x, y) = (2, 1)$. *Let* Γ *be the curve defined by the equation. Which of the following statements are correct?*

(A) $y(2) = 1$.

(B) $y'(2) = 2$.

(C) *The tangent line of* Γ *passing through* $(2, 1)$ *is given by* $y = -3x + 7$.

(D) *The normal line of* Γ *passing through* $(2, 1)$ *is given by* $y = \frac{1}{3}x + \frac{1}{3}$.

Answer: (A), (C), (D)
From the online test system §**2.2**.

Part II: Single answer questions. Select only ONE correct choice from a list of four choices.

Problem 7. (5 points) *Consider the function*

$$f(x) = \begin{cases} \frac{x^2-7x+12}{x-3}, & \text{if } x \neq 3; \\ 1, & \text{if } x = 3. \end{cases}$$

Then at $x = 3$

(A) f *is continuous.*

(B) f *has a removable discontinuity.*

(C) f *has a jump discontinuity.*

(D) f *has an infinite discontinuity.*

Answer: (B)
Modified from the textbook §**1.8** P. 26.

Problem 8. (5 points) *Let* m, b *be the constants such that the function*

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 2; \\ mx + b, & \text{if } x > 2. \end{cases}$$

is differentiable everywhere. Then $(m, b) =$

(A) $(1, 2)$.

(B) $(2, 0)$.

(C) $(4, -4)$.

(D) $(4, 2)$.

Answer: (C)
From the textbook §**2.3** P. 115.

Problem 9. (6 points) *Let A, B be the constants such that the function $y = A\sin(x) + B\cos(x)$ satisfies the differential equation $y'' + y' - 2y = \sin(x)$. Then $(A, B) =$*

(A) $(1, 0)$.

(B) $(2, -1)$.

(C) $(\frac{3}{5}, \frac{-2}{5})$.

(D) $(\frac{-3}{10}, \frac{-1}{10})$.

Answer: (D)

From the textbook §**2.4** P. 63.

Problem 10. (6 points) *Consider the function*

$$f(t) = \sqrt{\frac{1+\sin(t)}{1+\cos(t)}}.$$

Then $f'(0) =$

(A) 1.

(B) $2^{\frac{1}{2}}$.

(C) $2^{\frac{-1}{2}}$.

(D) $2^{\frac{-3}{2}}$.

Answer: (D)

From the textbook §**2.5** P. 26.

Problem 11. (6 points) *Let f be a differentiable function satisfying $f(1) = 10$, and $f'(x) \geq 2$ for all $1 \leq x \leq 4$. Which of the following values for $f(4)$ is impossible?*

(A) 15.

(B) 16.

(C) 2021.

(D) 5201314.

Answer: (A)
From the textbook §**3.2** P. 29.

Problem 12. (6 points) *Consider the function* $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = x^{2021} + x^{51} + x + 1.$$

(A) f *has a local maximum.*

(B) f *has a local minimum.*

(C) f *has a saddle point.*

(D) f *has neither a local maximum nor a local minimum.*

Answer: (D)
Modified from the textbook §**3.1** P. 72.

Problem 13. (6 points) *The linear approximated value of* $\sqrt{0.99}$ *is*

(A) $1 - 0.003$.

(B) $1 - 0.004$.

(C) $1 - 0.005$.

(D) $1 - 0.006$.

Answer: (C)
From the online test system §**3.1**.

Problem 14. (6 points) *Consider the function* $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{x^2 + 1}{x^2 - 1}.$$

How many horizontal/vertical/slant asymptotes does the graph of f *have?*

(A) 0.

(B) 1.

(C) 2.

(D) 3.

Answer: (D)
From the online test system §**3.3**.

Problem 15. (6 points) *Consider the function* $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{x^2 + 1}{x^2 - 1}.$$

How many reflection points does the graph of f *have?*

(A) 0.

(B) 1.

(C) 2.

(D) 3.

Answer: (A)
From the online test system §**3.3**.