Calculus Exam (Group A)

I Multiple Answer Questions (複選題)

To get all points for each question, you must select ALL correct choices and NONE of incorrect choices. If you miss a correct choice or taking an incorrect choice, then you will lose 50% of the full points. For all other cases you will get zero points.

Problem 1. (10 points) *Consider the function* $F : \mathbb{R} \to \mathbb{R}$ *defined by*

$$F(x) = \int_x^{x^2} \frac{1}{t^2+4}\,\mathrm{d}t.$$

Which of the following statements are correct?

(A) $F(0) = 0$.

(B) $F(-x) = F(x)$ *for all* $x \in \mathbb{R}$.

(C) $\lim_{x\to\infty} F(x) = 0$.

(D) $\lim_{x\to\infty} xF(x) = 0$.

Answer: (A), (C).

Problem 2. (9 points) *Let* $f(x)$ *be the inverse function of* $g(x) = x + \sin(x)$ *for* $x \in [0, \pi]$. *Which of the following statements are correct?*

(A) $f(0) = 0$.

(B) $f(\pi) = \frac{\pi}{2}$.

(C) $f'(0) = 1$.

(D) $\int_0^\pi f(x)\,\mathrm{d}x = \frac{1}{2}\pi^2 - 2$.

Answer: (A), (D). Modified from the Online Test System §**6.1**.

Problem 3. (9 points) *A student is trying to prove*

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0$$

for any two positive integers m *and* n, *and writing the following proof. However, the student might not provide a valid proof. Please select the correct steps.*

(A) $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{1}{2}\left[\sin(mx-nx) + \sin(mx+nx)\right]\,\mathrm{d}x.$

(B) $\int_{-\pi}^{\pi} \frac{1}{2}\left[\sin(mx-nx) + \sin(mx+nx)\right]\,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} \left[\sin(mx-nx)\right]\,\mathrm{d}x + \frac{1}{2}\int_{-\pi}^{\pi} \left[\sin(mx+nx)\right]\,\mathrm{d}x.$

(C) $\frac{1}{2}\int_{-\pi}^{\pi}[\sin(mx-nx)]\,\mathrm{d}x+\frac{1}{2}\int_{-\pi}^{\pi}[\sin(mx+nx)]\,\mathrm{d}x=\frac{1}{2}\left[\frac{\cos(mx-nx)}{m-n}\right]_{-\pi}^{\pi}+\frac{1}{2}\left[\frac{\cos(mx+nx)}{m+n}\right]_{-\pi}^{\pi}.$

(D) *Since the cosine function is even, we have*

$$\begin{aligned}&\frac{1}{2}\left[\frac{\cos(mx-nx)}{m-n}\right]_{-\pi}^{\pi}+\frac{1}{2}\left[\frac{\cos(mx+nx)}{m+n}\right]_{-\pi}^{\pi}\\&=\frac{1}{2}\left[\frac{\cos(m\pi-n\pi)}{m-n}-\frac{\cos(-m\pi+n\pi)}{m-n}\right]+\frac{1}{2}\left[\frac{\cos(m\pi+n\pi)}{m+n}-\frac{\cos(-m\pi-n\pi)}{m+n}\right]\\&=0.\end{aligned}$$

Answer: (A) (B). From the Textbook §**7.2** Exercise 75.

Problem 4. (9 points) *Which of the following correctly simplify*

$$\int\frac{x\,\mathrm{d}x}{\sqrt{x^2+a^2}},\quad \textit{where } a>0?$$

(A) $\int\frac{x\,\mathrm{d}x}{\sqrt{x^2+a^2}}\overset{y=x^2}{=}\int\frac{\mathrm{d}y}{\sqrt{y+a^2}}\overset{z=y+a^2}{=}\int z^{-\frac{1}{2}}\,\mathrm{d}z=2z^{\frac{1}{2}}+C=2\sqrt{x^2+a^2}+C.$

(B) $\int\frac{x\,\mathrm{d}x}{\sqrt{x^2+a^2}}\overset{x=a\tan(\theta)}{=}\int\frac{\tan(\theta)\sec^2(\theta)\,\mathrm{d}\theta}{a\sec(\theta)}=\frac{1}{a}\sec(\theta)+C=\frac{1}{a}\sqrt{1+\tan^2(\theta)}+C=\frac{\sqrt{a^2+x^2}}{a^2}+C.$

(C) $\int\frac{x\,\mathrm{d}x}{\sqrt{x^2+a^2}}\overset{x=a\sinh(t)}{=}\int\frac{(a\sinh(t))\,(a\cosh(t)\,\mathrm{d}t)}{a\cosh(t)}=a\cosh(t)+C=a\cosh\left(\sinh^{-1}\left(\frac{x}{a}\right)\right)+C.$

(D) $\int\frac{x\,\mathrm{d}x}{\sqrt{x^2+a^2}}\overset{y=x^2}{=}\int\frac{\mathrm{d}y}{2\sqrt{y+a^2}}\overset{z=y+a^2}{=}\int\frac{z^{-\frac{1}{2}}}{2}\,\mathrm{d}z=z^{\frac{1}{2}}+C=\sqrt{x^2+a^2}+C.$

Answer: (C) (D). From the Textbook §**7.3** Exercise 37.

Problem 5. (9 points) *If $f(t)$ is continuous for $t\geq 0$, the* Laplace transform *of f is the function F defined by*

$$F(s)=\int_0^{\infty}f(t)\,e^{-st}\,\mathrm{d}t$$

and the domain of F is the set consisting of all numbers s for which the integral converges. Which of the following function can be transformed by the Laplace transform?

(A) e^{e^t}.

(B) $e^{1+\sqrt{t}}$.

(C) $1+\sqrt{e^t}$.

(D) $1+\sqrt{1+\sqrt{t}}$.

Answer: (B) (C) (D). From the Textbook §**7.8** Exercise 85.

II Single Answer Questions (單選題)

Select only ONE correct choice from a list of four choices.

Problem 1. (6 points) *What is $f'(0)$, where*

$$f(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,\mathrm{d}u?$$

(A) -1.

(B) 1.

(C) 3.

(D) 5.

Answer: (A). From the Textbook §**4.3** P. 59.

Problem 2. (6 points) *What is the constant $c \in \mathbb{R}$ such that*

$$\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}\,\mathrm{d}x = c\int_0^{\pi} \frac{\sin(x)}{1+\cos^2(x)}\,\mathrm{d}x?$$

You may use the substitution $u = \pi - x$.

(A) π.

(B) $\frac{\pi}{2}$.

(C) π^2.

(D) $\frac{\pi^2}{4}$.

Answer: (B). From the Textbook §**4.5** P. 84.

Problem 3. (6 points) *What is the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = x$ around the y-axis?*

(A) $\frac{1}{2}\pi$.

(B) $\frac{3}{2}\pi$.

(C) $\frac{9}{2}\pi$.

(D) $\frac{27}{2}\pi$.

Answer: (D). From the Textbook §**5.3** P. 50.

Problem 4. (6 points) *Find the limit*

$$\lim_{x\to 1^+} x^{\frac{1}{1-x}}.$$

(A) 1.

(B) e.

(C) e^{-1}.

(D) -1.

Answer: (C). From the Textbook §**6.8** P. 61.

Problem 5. (6 points) *A student wants to inductively use the following reduction formula:*

$$\int (\ln x)^n \, dx = x (\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

to prove $\int (\ln x)^n \, dx = \sum_{m=0}^{n} (-1)^{n-m} \frac{n!}{m!} x (\ln x)^m + C$ *for* $n \in \mathbb{N}$ *. This students writes the proof as follows.*

1. *When* $n = 0$*, we have*

$$\int (\ln x)^0 \, dx = \int dx = x + C = (-1)^{0-0} \frac{0!}{0!} x (\ln x)^0 + C = \sum_{m=0}^{0} (-1)^{0-m} \frac{0!}{m!} x (\ln x)^m + C.$$

2. *Assume when* $n = k$*, we have*

$$\int (\ln x)^n \, dx = \int (\ln x)^k \, dx = \sum_{m=0}^{k} (-1)^{k-m} \frac{k!}{m!} x (\ln x)^m + C = \sum_{m=0}^{n} (-1)^{n-m} \frac{n!}{m!} x (\ln x)^m + C.$$

   Then, when __________*, we also have* $\int (\ln x)^n \, dx = \cdots = \sum_{m=0}^{n} (-1)^{n-m} \frac{n!}{m!} x (\ln x)^m + C$.

3. *Therefore, by mathematical induction,*

$$\int (\ln x)^n \, dx = \sum_{m=0}^{n} (-1)^{n-m} \frac{n!}{m!} x (\ln x)^m + C, \quad \text{for } n \in \mathbb{N}.$$

Which of the following formula should be put in the underline location so that the previous forms a valid inductive proof, where the dots represent a long computation and omits here.

(A) $n = k + 2$.

(B) $n = k + 1$.

(C) $n = k$.

(D) $n = k - 1$.

Answer: (B). From the Textbook §**7.1** Exercise 57.

Problem 6. (6 points) *Find the exact length of the curve*

$$y = \frac{2}{3} \left(x^2 - 1\right)^{\frac{3}{2}} \quad \text{for } 1 \le x \le 2.$$

(A) $\frac{3}{2}$.

(B) $-\dfrac{11}{3}$.

(C) $\dfrac{11}{3}$.

(D) $\dfrac{17}{3}$.

Answer: (C). From the Textbook §**8.1** Exercise 11.

Problem 7. (6 points) *Find the surface area obtained by rotating the curve* $y = \frac{x^3}{3}$, $0 \leq x \leq 1$, *about the* x*-axis.*

(A) $\dfrac{1}{12}$.

(B) $\dfrac{2}{9}\sqrt{2}\pi$.

(C) $\dfrac{\pi}{9}\left(2\sqrt{2}-1\right)$.

(D) $\dfrac{\pi}{63}$.

Answer: (C). From the Online Test System §**8.2**.

Problem 8. (6 points) *Evaluate*

$$\int_0^1 \frac{x^4 + x^3 + x^2 + x + 1}{x^2 + 1}\,\mathrm{d}x.$$

(A) $\dfrac{5}{6} + \ln 2$.

(B) $2 + \dfrac{\pi}{4}$.

(C) $2 + \ln 2$.

(D) $\dfrac{5}{6} + \dfrac{\pi}{4}$.

Answer: (D). From the Online Test System §**7.3**.

Problem 9. (6 points) *Evaluate*

$$\int \frac{e^{\frac{3}{x}}}{x^2}\,\mathrm{d}x.$$

(A) $-\dfrac{1}{3}e^{\frac{3}{x}} + C$.

(B) $\dfrac{1}{3}e^{\frac{3}{x}} + C$.

(C) $-\dfrac{1}{3}e^{x} + C$

(D) $\dfrac{1}{3}e^{x} + C$.

Answer: (A). From the Textbook §**7.5** Exercise 40.