(k, d)-choosable of graphs

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A proper $k$-coloring of $G$ is a mapping $\phi : V(G) \rightarrow \{1, 2, \cdots, k\}$ such that $\phi(x) \neq \phi(y)$ for every $xy \in E(G)$. The chromatic number of $G$, denoted by $\chi(G)$, is the least number of colors in an proper vertex coloring of $G$.

It is well-known that if $G$ is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$. Every planar graph is 4-vertex-colorable (四色定理).

一个平面图$G$是3-点可染的，如果满足下列条件之一:

- $G$不含5-圈，同时两个3-圈的距离至少为2；
- $G$不含5-, 7-圈，同时两个三角形不相邻[J C T(B) 96 (2006) 958 – 963];
A graph $G$ is said to be $f$-choosable if, whenever we give lists of $f(x)$ colors to each vertex $v \in V(G)$, there exists a proper vertex coloring of $G$ where each vertex is colored with a color from its own list. If $f(x) = k$ for every vertex $x \in V(G)$, we say that $G$ is $k$-choosable. The choice number or the list chromatic number $\chi_{list}(G)$ is the smallest integer $k$ such that $G$ is $k$-choosable.

The concept of a list coloring was introduced by Vizing[Metody Diskret. Analiz, Novosibirsk 29 (1976) 3-10] and Erdős, Rubin and Taylor[Congr. Numer., 26 (1979), 125 – 157], respectively. It is obvious that $\chi_{list}(G) \leq \Delta + 1$. 

\[ \begin{align*} 
\{1,2\} & \quad \{1,3\} \\
\{2,3\} & \quad \{3,5\} \\
\{1,2\} & \quad \{1,2\} \\
\{1,3\} & \quad \{1,3\} \\
\{2,3\} & \quad \{2,3\} 
\end{align*} \]
C. Thomassen, Every planar graph is 5-choosable, J Comb Theory (B) 62 (1994) 180-181.

If $G$ is a plane graph with outer cycle $C$, $p_1$ and $p_2$ are two adjacent vertices on $C$, and $L$ is a list assignment for $G$ such that $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus V(P)$, $|L(p_1)| = |L(p_2)| = 1$ and $L(p_1) \neq L(p_2)$, then $G$ is $L$-colorable.

It implies that $\chi_{list}(G) \leq 5$ for any planar graph $G$. The result is best possible, see[M. Voigt, Disc Math 120 (1993) 215-219].
If $G$ is a plane graph with outer cycle $C$, $v_1, v_2 \in V(C)$ and $L$ is a list assignment for $G$ with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{v_1, v_2\}$, and $|L(v_1)| = |L(v_2)| = 2$, then $G$ is $L$-colorable.

If $G$ is a planar graph with list assignment $L$ that gives lists of size one or five to its vertices and the distance between any pair of vertices with lists of size one is at least 20780, then $G$ is $L$-colorable.

Every graph drawn in the plane so that the distance between every pair of crossings is at least $15$ is $5$-choosable.

Every graph embedded in a fixed surface with sufficiently large edge-width is $5$-choosable.
Let $e = xy$ be an edge of a graph $G = (V, E)$. To contract an edge $e$ of a graph $G$ is to delete the edge and then (if the edge is a link) identify its ends. A graph $H$ is a minor of a graph $G$ if $G$ has a subgraph contractible to $H$; $G$ is called $H$-minor free if $G$ does not have $H$ as a minor.

Let $G$ be a $K_5$-minor-free graph and let $L$ be a list assignment of $G$ such that $|L(v)| \geq 5$ for every vertex $v \in V(G)$. Suppose that $H$ is a subgraph of $G$ isomorphic to $K_2$ or $K_3$ and suppose that $\lambda$ is an $L$-coloring of $H$. Then $\lambda$ can be extended to an $L$-coloring of $G$.
Let $G$ be a graph and $\sigma : E(G) \rightarrow \{1, -1\}$ be a mapping. The pair $(G, \sigma)$ is called a signed graph, and $\sigma$ is called a signature of $G$. An edge $e$ is positive (or negative) if $\sigma(e) = 1$ (or $\sigma(e) = -1$). Proper coloring means that for any edge $uv \in E(G)$, $f(u) \neq \sigma(uv)f(v)$.

Every signed planar graph is 5-choosable and that there is a signed planar graph which is not 4-choosable while the unsigned graph is 4-choosable. For each $k \in \{3, 4, 5, 6\}$, every signed planar graph without circuits of length $k$ is 4-choosable. Furthermore, every signed planar graph without circuits of length 3 and of length 4 is 3-choosable. They construct a signed planar graph with girth 4 which is not 3-choosable but the unsigned graph is 3-choosable.

If $G$ is a bipartite planar graph, then $\chi_{list}(G) \leq 3$.


If $G$ is a planar graph of girth $g \geq 5$, then $\chi_{list}(G) \leq 3$.


李相文, On 3-choosable planar graphs of girth at least 4, Discrete Mathematics 309 (2009) 2424-2431

If a planar graph $G$ has neither intersecting 4-cycles nor a 5-cycle intersecting with any 4-cycle, then $G$ is 3-choosable.

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Let $G$ be a planar graph. Then $G$ is 3-choosable if one of the following conditions holds.

- $G$ contains no cycles of length in $\{4, 6, 8, 9\}$; [L Shen, Y Q Wang, Information Processing Letters 104 (2007) 146 - 151]
- $G$ contains no cycles of length in $\{4, 6, 7, 9\}$; [Y Q Wang, H J Lu, M Chen, Information Processing Letters 105 (2008) 206 - 211]
- $G$ contains no cycles of length in $\{4, 5, 8, 9\}$; [Y Q Wang, H J Lu, M Chen, Disc Math, 310(2010),147-158]
- $G$ contains no cycles of length in $\{4, 7, 8, 9\}$. [Y Q Wang, Q Wu, L Shen, Disc Appl Math, 159(2011), 232-239]
Let $G$ be a planar graph. Then $G$ is 3-choosable if one of the following conditions holds.

- $G$ contains no cycles of length 4 and 5 and every two triangles has distance at least 4, or $G$ contains no cycles of length 4, 5 and 6 and any two triangles has distance at least 3; [王维凡等, Disc Math 306(2006) 573 – 579]
- $G$ contains no cycle of length at most 10 with a chord; [W F Wang, Taiwanese J Math, 11(2007) 179-186]
- $G$ contains no cycles of length 5, 6 and 7 and any two triangles has distance at least 3, or $G$ contains no cycles of length 5, 6 and 8 and any two triangles has distance at least 2; [H H Zhang Z R Sun, Information Processing Letters 107 (2008) 102 – 106]

If $G$ is a planar graph such that the distance between any two ($\leq 4$)-cycles is at least 26, then $G$ is 3-choosable.

Every planar graph $G$ without cycles of lengths 4 to 8 is 3-choosable.

If $G$ is a planar graph without 4-cycles, then $\chi_{list}(G) \leq 4$.


If $G$ is a planar graph without 5-cycles, then $\chi_{list}(G) \leq 4$.


If $G$ is a planar graph without 6-cycles, then $G$ is 3-degenerate and it follows that $\chi_{list}(G) \leq 4$. 

If $G$ is a planar graph without 4-cycles, then $\chi_{\text{list}}(G) \leq 4$.


If $G$ is a planar graph without 5-cycles, then $\chi_{\text{list}}(G) \leq 4$.


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B. Farzad, Planar graphs without 7-cycles are 4-choosable, SIAM J Discrete Math 23 (2009), 1179 – 1199.

If $G$ is a planar graph without 7-cycles, then $\chi_{list}(G) \leq 4$.


If $G$ is a planar graph without intersecting 3-cycles (that is, every vertex is incident with at most one 3-cycle), then $\chi_{list}(G) \leq 4$.


All planar graphs without triangular 4-cycles are 4-choosable.
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All planar graphs without triangular 4-cycles are 4-choosable.
R Y Xu and W, A sufficient condition for a planar graph to be 4-choosable, Discrete Applied Mathematics 224 (2017) 120 – 122.

Let $G$ be a planar graph. If every 5-cycle of $G$ is not adjacent simultaneously to 3-cycles and 4-cycles, then $G$ is 4-choosable.

D Q Hu and W, Planar graphs without intersecting 5-cycles are 4-choosable, Discrete Mathematics 340 (2017) 1788 – 1792

Planar graphs without intersecting 5-cycles are 4-choosable.


Planar graphs without chordal 6-cycles are 4-choosable.
A graph $G$ is said to be $(k, s)$-choosable ($k \geq s$) if for each list assignment $L$ satisfying $|L(v)| \geq k$ for each vertex $v$ and $|L(x) \cap L(y)| \leq s$ for each edge $xy$, $G$ has an $L$-coloring (即在列表染色中增加了列表的要求：任意相邻两点的列表至多有$s$种颜色相同). Let $\chi_l(G, s)$ denote the minimum $k$ such that $G$ is $L$-colorable for each $s$-separated $k$-list $L$.

Kratochvíl等J. Graph Theory, 27 (1998), 43-49.

(1) For positive integers $s, n$ with $s \leq n$,

$$\sqrt{\frac{1}{2}sn} \leq \chi_l(K_n, s) \leq \sqrt{2esn}.$$  

(2) $\chi_l(G, s) \leq \sqrt{2es(\Delta(G) - 1)}$.


(4) Every triangle-free planar graph is $(3, 1)$-choosable.
Thomassen proved that planar graphs are 5-choosable and hence they are \((5, d)\)-choosable for all \(d\). Voigt constructed a non-4-choosable planar graph and there are also examples of non-(4, 3)-choosable planar graphs. Škrekovski observed that there are examples of triangle-free planar graphs that are not \((3, 2)\)-choosable.

**Conjecture 1**

Every planar graph is \((4, 2)\)-choosable.

It is proved for all planar graphs without chorded \(l\)-cycles, for each \(l \in \{5, 6, 7\}\) [Graphs and Combinatorics (2017) 33:751 – 787].

**Conjecture 2**

Every planar graph is \((3, 1)\)-choosable

Every planar graph $G$ is $(3, 1)$-choosable if any $i$-cycle is not adjacent to a $j$-cycle, where $5 \leq i \leq 6$ and $5 \leq j \leq 7$. 

Let $G = (V, E, F)$ be a counterexample to the result with the fewest vertices. Then

(a) $G$ has a $2^-$-vertex, or
(b) $G$ contains one of the configurations $(C1)$-$(C19)$.
证明思路

(C8)

(C9)

(C10)

(C11)

(C12)

(C13)

(C14)

(C15)

(C16)

(C17)

(C18)

(C19)

(k, d)-choosable of graphs
证明思路

Euler公式

\[ |V| - |E| + |F| = 2 \]

根据 \( \sum_{v \in V(G)} d(v) = 2|E| \), \( \sum_{f \in F(G)} d(f) = 2|E| \), 我们得到

\[ \sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0. \]  \hspace{1cm} (1) \]

\[ \sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (d(f) - 4) = -4(|V| - |E| + |F|) = -8 < 0. \]  \hspace{1cm} (2)
By Euler’s formula, we have the following formula.

\[
\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = 6(|E| - |V| - |F|) = -12 < 0.
\]

Now we assign an initial charge \( \mu(z) \) to each \( z \in V(G) \cup F(G) \) by letting \( \mu(v) = 2d(v) - 6 \) for \( v \in V(G) \), \( \mu(f) = d(f) - 6 \) for \( f \in F(G) \). Thus we have

\[
\sum_{z \in V(G) \cup F(G)} \mu(z) < 0.
\]
We define the following two rounds of discharging rules. The first round contains (R1)-(R4), which are called vertex rules.

R1. Suppose $d(v) = 4$.

(R1A) If $v$ is incident with exactly one $5^-$-face $f$, then $v$ sends 2 to $f$.

(R1B) Suppose that $f_{5^-}(v) = 2$. If $v$ is incident with a 3-face $f_1$ and a 4-face $f_2$ such that $f_2$ is incident with at least three $4^+$-vertices, then $v$ sends $\frac{4}{3}$ to $f_1$ and $\frac{2}{3}$ to $f_2$. Otherwise, $v$ sends 1 to each incident $5^-$-face.

(R1C) If $f_{5^-}(v) = 3$, then $v$ sends $\frac{2}{3}$ to each incident $5^-$-face.

R2. Suppose $d(v) = 5$.

(R2A) If $f_{5^-}(v) \leq 2$, then $v$ sends 2 to each incident $5^-$-face.

(R2B) If $f_{5^-}(v) = 3$ and $f_3(v) \leq 1$, then $v$ sends $\frac{3}{2}$ to its incident 3-face (if exists) and $\frac{5}{4}$ to each incident $i$-face, where $i \in \{4, 5\}$.

(R2C) If $f_{5^-}(v) = 3$ and $f_3(v) = 2$, then $v$ sends $\frac{3}{2}$ to each incident 3-face and 1 to its incident $i$-face, where $i \in \{4, 5\}$.

(R2D) Suppose $f_3(v) = 3$. Assume that the faces incident with $v$ are $f_1, f_2, \cdots, f_5$ in clockwise order such that $f_1, f_2$ are 3-faces. If $f_3$ or $f_5$ is a 3-face, then $v$ sends $\frac{4}{3}$ to each incident 3-face. Otherwise $f_4$ is a 3-face. If $f_4$ is bad and at most one of $f_1, f_2$ is bad, then $v$ sends $\frac{3}{2}$ to $f_4$ and $\frac{5}{4}$ to $f_i$ ($i = 1, 2$). Otherwise $v$ sends 1 to $f_4$ and $\frac{3}{2}$ to $f_i$ ($i = 1, 2$).
**R3.** Suppose that \( d(v) = 6, 7, 8 \). If \( f_{5-}(v) < \lfloor \frac{3d(v)}{4} \rfloor \), then \( v \) sends \( 2 \) to each incident \( 5^- \)-face. Otherwise \( v \) sends \( \frac{3}{2} \) to each incident \( 5^- \)-face.

**R4.** If \( d(v) \geq 9 \), then \( v \) sends \( 2 \) to each incident \( 5^- \)-face.

The second round contains (R5), which is called the **face rule**.

**R5.** Let \( f \) be a \( d \)-face where \( d \geq 7 \). Let \( f_0, f_1, f_2, \ldots, f_{d-1} \) be the faces adjacent to \( f \) in clockwise order, and let \( v_0, v_1, \ldots, v_{d-1} \) be the vertices incident with \( f \) in clockwise order such that \( v_i \) is incident with \( f_i \) and \( f_{i+1} \), where the subscripts are taken modulo \( d \) here.

*(R5A)* \( f \) sends \( \frac{d(f)-6}{d(f)} \) to \( f_i \) for any \( i \) (\( 0 \leq i < d \)).

*(R5B)* Suppose that \( f_i \) is hungry for some \( i(0 \leq i < d) \). (1) If \( f_{i+1} \) is not hungry and \( d(v_i) \leq 4 \), then \( f_{i+1} \) sends \( \frac{d(f)-6}{d(f)} \) to \( f_i \). (2) If \( f_{i-1} \) is not hungry, \( d(v_{i-1}) \leq 4 \), and either \( d(v_{i-2}) \geq 5 \) or \( f_{i-2} \) is not hungry, then \( f_{i-1} \) also sends \( \frac{d(f)-6}{d(f)} \) to \( f_i \).
(k, s)-可选的

如果给图G的所有点v都分配一个颜色集合L(v)(也叫色列表)且G存在一个正常的点染色φ 使得对每个点v ∈ V(G) 都有φ(v) ∈ L(v),则称图G是L-可染的. 如果对任何的色分配L, 满足|L(v)| ≥ k(∀v ∈ V(G))和|L(x) ∪ L(y)| ≥ s(∀xy ∈ E(G)), G是L-可染的，则称G是(k, s)-可选的.

(1) 均匀染色(equitable coloring):任何两个不同的颜色所染的顶点数至多差1；
(2) 无圈点染色(acyclic coloring): 任何两个不同的颜色所染的点集合所导出的子图是一个森林；
(3) 线性染色(linear coloring): 任何两个不同的颜色所染的点集合所导出的子图是一个线性森林；
(4) (p,q)-标号((p,q)-labelling): 相邻的顶点的颜色至少差p，距离为2的两个点的颜色至少差q;
(5) 邻点可区别的点染色(adjacent vertex distinguishing vertex coloring):任何相邻的两个点所对应的邻域的染色集合不同；
(6) 邻和可区别的点染色(adjacent sum distinguishing vertex coloring): 任何相邻的两个点所对应的邻域的颜色之和不相等；
(7) r-色调染色(r-hued coloring):度数为d的顶点邻域至少出现\( \min\{d, r\} \)种颜色;
(8) 邻域r-限制染色(Neighborhood r-bounded coloring): 每个点的同色邻点数不得超过r个
两个分数方面的染色

圆染色(circular coloring): 用颜色1, 2, ..., k给图的每个点染色f. 如果对任意的uv ∈ E(G),
有d ≤ |f(u) − f(v)| ≤ k − d(k ≥ 2d ≥ 1)，则称G存在(k, d)-圆染色. 最小的k称为G的圆色数χ^C(G).
等价定义：给一个周长为k的圆环L(k ≥ 2), 图G的每个点对应于L上长度为1的开弧. 如果两个点相邻, 它们对应的弧不交, 我们就说G是k-圆可染的. 最小的k称为G的圆色数.
For any graph G, χ(G) − 1 < χ^C(G) ≤ χ(G).

分数染色(fractional coloring): 用k种颜色给图的每个点染d种颜色. 如果任何相邻的两个点所染颜色的集合不交，则称G存在(k, d)-分数染色. 最小的k/d称为G的分数色数.

最近有个结果引起了不少的轰动：every planar graph has a (9, 2)-fractional coloring.
不要求正常情况下的染色

(k, d)*-染色: 用k种颜色去染图的点使得染同色的点集合的导出子图的最大度至多为d;

点荫度(vertex arboricity): 用k种颜色去染图的点使得染同色的点集合的导出子图是一个森林，所用最少的颜色数称为图的点荫度va(G);

点线性荫度等(linear vertex arboricity): 用k种颜色去染图的点使得染同色的点集合的导出子图是一个线性森林，所用最少的颜色数k称为图的点线性荫度vla(G);

圆点荫度(circular vertex arboricity): 用k(≥ 2d)种颜色去染图的点使得对每个j(0 ≤ j ≤ k – 1), 染j, j + 1, ..., j + d – 1的所有点的导出子图是一个森林(下标按k取模运算), 最小的k/d成为G的圆点荫度;
组合情况下的新的染色

- 均匀点荫度: 同色点集合的导出子图是一个森林，而且任何两个颜色所染的点数至多差1；
- 邻和可区别的(p,q)-标号: 给图G一个正常的点染色使得任何相邻的两个点所对应的邻域的颜色之和至少差p, 任何距离为2的两个点所对应的邻域的颜色之和至少差q；
- 邻和可区别的点荫度: 相同的邻和所导出的子图为森林；
- ...

还有game coloring, cochromatic number, achromatic number, antimagic label等.
Every planar graph $G$ without cycles of lengths 4 to 8 is 3-choosable.

Given a list $L$ for a graph $G$, the vertex set of the auxiliary graph $H = H(G, L)$ is \{$(v, c) : v \in V(G)$ and $c \in L(v)$\}, and two distinct vertices $(v, c)$ and $(v', c')$ are adjacent in $H$ if and only if either $c = c'$ and $vv' \in E(G)$, or $v = v'$.

$G$ has an $L$-coloring if and only if the independence number of $H$ is $|V(G)|$. 

\[G \cong C_4\]
The definition of DP-coloring

Let $G$ be a graph. A cover of $G$ is a pair $(L, H)$, where $L$ is an assignment of pairwise disjoint sets to the vertices of $G$ and $H$ is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$, satisfying the following conditions.

1. For each $v \in V(G)$, $H[L(v)]$ is a complete graph.
2. For each $uv \in E(G)$, the edges between $L(u)$ and $L(v)$ form a matching (possibly empty).
3. For each distinct $u, v \in V(G)$ with $uv \notin E(G)$, no edges of $H$ connect $L(u)$ and $L(v)$.

An $(L, H)$-coloring of $G$ is an independent set $I \subseteq V(H)$ of size $|V(G)|$. The DP-chromatic number, $\chi_{DP}(G)$, is the minimum $k$ such that $G$ has an $(L, H)$-coloring for each choice of $(L, H)$ with $|L(v)| \geq k$ for all $v \in V(G)$. 
$f$ -painting game

Given a graph $G$ and a mapping $f : V(G) \rightarrow \mathbb{N}$. The $f$ -painting game on $G$ is played by two players: Lister and Painter. Initially, all vertices are uncoloured and each vertex $v$ has $f(v)$ tokens. In the $i$th step, Lister marks a non-empty subset $L_i$ of uncoloured vertices and takes away one token from each marked vertex. Painter chooses an independent set $X_i$ contained in $L_i$ and colours vertices in $X_i$ by colour $i$. If at the end of some step, there is an uncoloured vertex $v$ with no tokens left, then Lister wins the game. Otherwise, at some step, all vertices are coloured and Painter wins the game.
Suppose \( f : V(G) \to N \). We say \( G \) is \( f \)-paintable if Painter has a winning strategy in the \( f \)-painting game on \( G \). We say \( G \) is \( s \)-paintable for a positive integer \( s \) if \( G \) is \( f \)-paintable for the constant function \( f \equiv s \). The paint number \( \mathbb{F}_p(G) \) (also called the paintability and the on-line choice number) of \( G \) is the least integer \( s \) for which \( G \) is \( s \)-paintable.

Ming Han, Xuding Zhu, Locally planar graphs are 5-paintable, Discrete Mathematics 338 (2015) 1740 – 1749.

Every graph embedded in a fixed surface with sufficiently large edge-width is 5-paintable.


Locally planar graphs are 2-defective 4-paintable.
Adaptably $k$-coloring (适应点染色)

定义

Let $G$ be a graph, and let $F : E(G) \rightarrow N$ be a coloring of the edges of $G$ (not necessarily proper). A vertex $k$-coloring $c : V(G) \rightarrow \{1, \cdots , k\}$ of the vertices of $G$ is adapted to $F$ if for every $uv \in E(G)$, $c(u) \neq c(v)$ or $c(v) \neq F(uv)$. In other words, the same color never appears on an edge and both its endpoints. If there is an integer $k$ such that for any edge coloring $F$ of $G$, there exists a vertex $k$-coloring of $G$ adapted to $F$, we say that $G$ is adaptably $k$-colorable. The smallest $k$ such that $G$ is adaptably $k$-colorable is called the adaptable chromatic number of $G$, denoted by $\chi_{ad}(G)$.

显然: $\chi_{ad}(G) \leq \chi(G)$.

J Graph Theory 62: 127 – 138, 2009

Every $K_5$-minor-free graph is adaptably 4-choosable;
Every triangle-free planar graph is adaptably 3-choosable.
An edge $k$-coloring of a graph $G$ is a mapping $\psi$ from $E(G)$ to the set of colors $1, 2, \ldots, k$ such that any two adjacent edges have different colors. The edge chromatic number of a graph $G$, denoted by $\chi'(G)$, is the smallest integer $k$ such that $G$ has an edge $k$-coloring.

**Vizing’s Theorem, 1964**

For every graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

A graph $G$ is said to be of class 1 if $\chi'(G) = \Delta$, and of class 2 if $\chi'(G) = \Delta + 1$. 
Edge Coloring of Planar Graphs

Four Coloring Problem

For every planar graph $G$, $\chi(G) \leq 4 \iff$ For every simple 2-edge-connected 3-regular planar $G$, $\chi'(G) = 3$.

If $C_4$, $K_4$, the octahedron, and the icosahedron have one edge subdivided each, class 2 planar graphs are produced for $\Delta \in \{2, 3, 4, 5\}$. Vizing proved that every planar graph with $\Delta \geq 8$ is of class 1 and then posed the following conjecture.

Conjecture 1: 平面图边染色猜想

Every planar graph with $\Delta \geq 6$ is of class 1.

1 Critical graphs with given chromatic class, Diskret. Analiz. 5 (1965) 9 - 17.
Conjecture 1 is true for planar graphs.

- $\Delta = 7^a$ and $b$; 
- $(\Delta, g) \in \{(5, 4), (4, 5), (3, 8)\}$, where $g$ is the girth of $G^c$; 
- $\Delta = 6$ and any vertex is incident with at most three triangles$^d$; 
- $\Delta \geq 5$ and any vertex is incident with at most one triangle$^e$; 
- $\Delta = 6$ and $G$ contains no chordal $k$-cycles for some $k \in \{3, 4, 5, 6, 7\}$.$^f$

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$^a$L. M. Zhang, Graphs Combin. 16 (2000), 467-495.
$^b$Sanders and Y. Zhao J. Combin. Theory Ser B 83 (2001), 202-212
$^c$Fiorini and R.J. Wilson, Research Notes in Mathematics, 16, 1977
$^e$陈永珠,王维凡,浙江师范大学学报，30:4(2007),416-420
$^f$倪伟平, 南京师大学报, 34:3(2011), 19-24

A graph $G$ is said to be edge $f$-choosable if, whenever we give lists $A_x$ of $f(x)$ colors to each element $x \in E(G)$, there exists a proper edge coloring of $G$ where each edge is colored with a color from its own list. If $f(x) = k$ for every element $x \in E(G)$, we said $G$ is edge $k$-choosable. The list edge chromatic number $\chi'_{list}(G)$ is the smallest integer $k$ such that $G$ is edge $k$-choosable.

**Conjecture 3.** Every graph satisfies $\chi'_{list} \leq \Delta + 1$.


**Conjecture 4.** For any graph $G$, $\chi'_{list}(G) = \chi'(G)$ and $\chi''_{list}(G) = \chi''(G)$.
### Some results on a planar graph with $\chi'_{list} \leq \Delta + 1$

- $\Delta \geq 9^a$, $\Delta = 8^b$ and $\Delta \leq 4^c$;
- $\Delta \geq 7$且没有$(1)$ 弦$7$-圈$^d$, 或$(2)$ 弦$6$-圈$^e$;
- $\Delta \geq 6$且不满足：$(1)$ 相邻$3$-圈$^f$, 或$(2)$ $3$-圈与$5$-圈相邻$^g$, 或$(3)$ 弦$5$-圈$^h$, 或$(4)$ 弦$6$-圈$^i$;
- $\Delta \geq 5$且没有：$(1)$ 弦$5$-圈和弦$4$-圈, 或$(2)$弦$5$-圈和弦$6$-圈, 或$(3)$ $3$-圈$^j$, 或$(4)$ $4$-圈$^k$, 或$(4)$ $5$-圈$^l$

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$a$ Borodin, Mat. Zametki 48 (6) (1990) 22 – 28

$b$ SIAM J. DISCRETE MATH, 29:3(2015), 1735 – 1763

$c$ J Graph Theory, 32(1999) 250-262.


$f$ Disc Math 309 (2009) 77 – 84

$g$ Disc Math 313 (2013) 575 – 580


$k$ Discrete Mathematics 308(2008) 5789 – 5794

## List Edge Coloring of Planar Graphs

**Planar graphs on** $\chi'_{list} = \Delta$

- $\Delta \geq 12$, or $\Delta \geq 7$ and $g \geq 4$, or $\Delta \geq 5$ and $g \geq 5$, or $\Delta \geq 4$ and $g \geq 6$, or $\Delta \geq 3$ and $g \geq 10$;
- $(\Delta, k) \in \{(7, 4), (6, 5), (5, 8), (4, 14)\}$, where $k$ satisfies that $G$ has no cycle of length from 4 to $k$, where $k \geq 4$.
- $\Delta \geq 8$ and $G$ contains no chordal 5-cycles;
- $\Delta \geq 8$ and $G$ contains no adjacent 4-cycles;
- $\Delta \geq 8$ 且 3-圈和4-圈不邻
- $\Delta \geq 8$ 且 3-圈和5-圈不邻, or $\Delta \geq 7$ 且 两个4-圈不邻
- $\Delta \geq 7$ and any 4-cycle is not adjacent to 4- cycles
- $\Delta \geq 6$ 且 没有4-圈和6-圈, or $\Delta \geq 7$ 且 没有5-圈和6-圈

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*c* J Comb Optim (2016) 32:188 – 197  
*d* J Comb Optim (2016) 31:1013 – 1022  
A planar graph $G$ is edge-$(\Delta(G) + 1)$-choosable if any 4-cycle is not adjacent to a 3-cycle.

If $G$ is a planar graph with $\Delta(G) \geq 6$ and has no adjacent 4-cycles, then $\chi'_{list}(G) \leq \Delta(G) + 1$.

If $G$ is a planar graph without 6-cycles with two chords, then $G$ is edge-$k$-choosable, where $k = \max\{7, \Delta(G) + 1\}$, and is edge-$t$-choosable, where $t = \max\{9, \Delta(G)\}$.

Let $G$ be a planar graph in which contains no 6-cycles with three chords or $G$ be a $f_5$-free planar graph. Then $G$ is edge-$k$-choosable, where $k = \max\{8, \Delta(G) + 1\}$, and is edge-$t$-choosable, where $t = \max\{10, \Delta(G)\}$.
A well-known conjecture

**Conjecture.** For any simple graph $G$, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$.

**Theorem**

*For a planar graph $G$, $\chi''(G) \leq \Delta + 2$, if one of the following conditions hold.*

- $\Delta \leq 5$ or $\Delta \geq 7$;
- $\Delta = 6$ and $v_5^4 + 2(v_5^5 + v_6^4) + 3v_6^5 + 4v_6^6 + < 24$, where $v_n^k$ represents the number of vertices of degree $n$ which lie on $k$ distinct 3-cycles; [Graphs and Comb. 30(2014), 377-388]
- $\Delta = 6$ and without 4-, 5-, or 6-cycles with chords [Hou and Liu]
- $\Delta = 6$ and two cycles of length at most 5 are not adjacent. [Wu and Fang]
Conjecture: For any planar graph $G$ with $\Delta \geq 5$, $\chi''(G) = \Delta + 1$.

对平面图$G$, $\chi''(G) = \Delta + 1$如果下列条件之一成立:

1. $\Delta \geq 14, 12, 11, 10$ and finally 9;
2. $\Delta \geq 8$ and for every vertex $x \in V(G)$, there is an integer $k \in \{3, 4, 5, 6, 7, 8\}$ such that $x$ is incident with at most one cycle of length $k$;
3. $\Delta \geq 8$ and for each vertex $x$, there are two integers $i, j \in \{3, 4, 5, 6\}$ such that any two cycles of length $i$ and $j$, which contain $x$, are not adjacent;
4. $\Delta \geq 8$ and $G$ is an $F_5$-free planar graph;
5. $\Delta \geq 8$ and $G$ contains no 5-cycles with two chords;
6. $\Delta \geq 8$ and $G$ contains no adjacent chordal 5-cycles;
7. $\Delta \geq 8$ and $G$ contains no adjacent chordal 7-cycles;
8. $\Delta \geq 8$ and $G$ contains no 6-cycles with two chords or adjacent chordal 6-cycles;
9. $\Delta \geq 8$ and $G$ contains no 7-cycles with three chords;
10. $\Delta \geq 7$ and for every vertex $x \in V(G)$, there is an integer $k \in \{3, 4, 5, 6, 7, 8\}$ such that $x$ is incident with no cycles of length $k$;
11. $\Delta \geq 7$ and every vertex $v$ has an integer $k_v \in \{3, 4, 5, 6\}$, such that $v$ is not in any $k_v$-cycle;
If the following conditions hold, then for a planar graph $G$ we have $\chi''(G) = \Delta + 1$:

12. $\Delta \geq 7$ and $G$ contains no intersecting 3-cycles, or adjacent 4-cycles, adjacent 5-cycles, or intersecting 6-cycles;

13. $\Delta \geq 7$ and $G$ contains no chordal $i$-cycles ($i = 5, 6, \text{ or } 7$);

14. $\Delta \geq 7$ and no 3-cycle is adjacent to a cycle of length less than 6;

15. $\Delta \geq 6$ and $G$ contains no 5-cycles and 6-cycles, or $\Delta \geq 5$ and $G$ contains no 4-cycles and 6-cycles;

16. $\Delta(G) \geq 6$, $G$ contains no intersecting 4-cycles and $G$ contains no intersecting 3-cycles, or 5-cycles, or 6-cycles;

17. $\Delta \geq 6$ and $G$ contains no 4-cycles;

18. $\Delta \geq 6$ and $G$ contains no adjacent $4^-$-cycles;

19. $(\Delta, g) \in \{(7, 4), (5, 5), (4, 6), (3, 10)\}$, where $g$ is the girth of $G$;

20. $(\Delta, k) \in \{(7, 4), (6, 5), (5, 7), (4, 14)\}$, where $G$ has no cycle of length from 4 to $k$, where $k \geq 4$;

21. $(\Delta, k) \in \{(6, 4), (5, 5), (4, 11)\}$, where $G$ contains no intersecting 3-cycles and $G$ has no cycle of length from 4 to $k$. 
A graph $G$ is said to be **total $f$-choosable** if, whenever we give lists $A_x$ of $f(x)$ colors to each element $x \in V(G) \cup E(G)$, there exists a proper total coloring of $G$ where each element is colored with a color from its own list. If $f(x) = k$ for every element $x \in V(G) \cup E(G)$, we said $G$ is total $k$-choosable. The list total chromatic number $\chi''_{list}(G)$ is the smallest integer $k$ such that $G$ is total $k$-choosable.


**Conjecture 4.** For any graph $G$, $\chi'_{list}(G) = \chi'(G)$ and $\chi''_{list}(G) = \chi''(G)$.

此猜想对二分图是成立的。
## List total coloring of Planar Graphs

**Some results on a planar graph with** $\chi''_{\text{list}}(G) \leq \Delta + 2$

- $\Delta \geq 9^a$;
- $\Delta \geq 7$ 且 (1) 没有弦7-圈$^b$, 或 (2) $F_5$-free $^c$, 或 (3) 每个3-圈至多与其他一个3-圈相邻$^d$;
- $\Delta \geq 6$ 且不满足：(1) 弦6-圈$^e$, 或 (2) 3-圈与5-圈相邻$^f$, 或 (3) 3-圈与4-圈相邻$^g$;
- $\Delta \geq 5$ 且没有：(1) 弦5-圈和弦4-圈, 或 (2) 弦5-圈和弦6-圈, 或 (3) 3-圈或4-圈$^h$, 或 (4) 5-圈$^i$

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^c J Comb Optim, to appear.
^d 山大学报,2009年10期
^f Disc Math 313 (2013) 575 – 580
^g Discrete Mathematics 311 (2011) 2158 – 2163
^h LNCS 4489 (2007) 320 – 328
# List total coloring of Planar Graphs

<table>
<thead>
<tr>
<th>Planar graphs on $\chi''_{list}(G) = \Delta + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \geq 12$; $^a$</td>
</tr>
<tr>
<td>$(\Delta, k) \in {(7, 4), (6, 5), (5, 8), (4, 14)}$, where $k$ satisfies that $G$ has no cycle of length from 4 to $k$, where $k \geq 4$. $^b$</td>
</tr>
<tr>
<td>$\Delta \geq 8$ and $G$ contains no chordal 5-cycles; $^c$</td>
</tr>
<tr>
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</tr>
</tbody>
</table>

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$c$ J Comb Optim (2016) 32:188 – 197  
$d$ J Comb Optim (2016) 31:1013 – 1022  
(1) 证明满足如下条件的平面图的列表色数至多为4，或列表点
荫度至多为2：4-圈不相交、6-圈不交、不含弦5-圈、3圈
与5-圈不相邻等。

(2) 最近我们证明了：Δ ≥ 7且7-圈至多有2条弦或6-圈至多
有2条弦的平面图的列表边色数至多为Δ + 1。这样我们还可
以考虑如下条件：(1) Δ ≥ 7且5-圈至多有1条弦；(2) Δ ≥ 7
且弦k-圈不相邻(k = 4, 5, 6, 7); (3) Δ ≥ 6且弦k-圈不相
交(k = 4, 5, 6, 7)。当然，列表边色数等于最大度、或列表线
性荫度、或列表全色数的条件可以更多。

(3) 证明：(1) 最大度为8的平面图的列表全色数至多为10; (2)
最大度为11的平面图的列表全色数等于12; (3) 考虑relaxed, separated, different的情况。

(4) 证明：最大度为6的图或平面图的全色数至多是8。

(5) 证明：最大度为7的图的线性荫度是4;
进一步研究和思考的问题

(6) 已知1-平面图的点色数至多为6，无圈点色数≤ 20，那么它的列表无圈点色数、无圈列表边色数的上界是多少？对NIC-和IC-平面图的结果又是多少？

(7) 已知Δ ≥ 9的平面图的均匀色数≤ Δ, 那么1-平面图的最大度至少为多少的时候也有此结果呢？列表均匀色数呢？

(8) 最近我们证明了：(1) 对任意的k ≥ 12, 所有的平面图都是可均匀k-边染色的; (2) 对任意的k ≥ 21, 所有的1-平面图都是可均匀k-边染色的。那么对嵌入到欧拉示性数小于0的曲面上的图，它的均匀边色数又是一个什么结果？

(9) 2016年在European Journal of Combinatorics上发表一篇题目为“Choosability in signed planar graphs”的文章，我们完全可以考虑符号图的其它染色。

(10) 把以上所得到结果的条件用于其它的染色，如无圈点染色、无圈边染色、无圈全染色、均匀点染色、均匀边染色、均匀点荫度、邻点可区别的各种染色等，但是要注意结果的包含关系；
谢谢各位的聆听！