On the $\alpha$-distance spectral radius

Bo Zhou

South China Normal University

August 20, 2019
In geometry, we consider $n$ points in a plane and the distances between two points is the Euclidean distance. We may form a distance matrix with $(i, j)$ entry to be the Euclidean distance between the $i$-th and the $j$-th points for $1 \leq i, j \leq n$.


G. Young, A. Householder, Discussion of a set of points in terms of their mutual distances, Psychometrika 3 (1938) 19–22.
Hakimi and Yau considered a graph $G$ on $n$ vertices with weights $w_{uv}$ for each edge $uv$. The distance matrix $D = (d_{uv})$ is an $n \times n$ matrix whose entry $d_{uv}$ is the shortest weighted distance between vertices $u$ and $v$.

For a given real symmetric $n \times n$ matrix $D = (d_{ij})$ such that $d_{ii} = 0$ and $0 \leq d_{ij} \leq d_{ik} + d_{kj}$ for any $1 \leq i, j, k \leq n$, they consider the problem whether there is there a graph $G$ for which $D$ is its distance matrix.

Particularly, they also considered the case $w_{uv} = 1$ for $uv \in E(G)$.


We consider graph-theoretical distance and facts on spectral properties distance matrix. Let $G$ be a connected graph with vertex set $V(G)$. The distance between vertices $u, v$ is defined as the number of edges of a shortest path between them in $G$, denoted by $d_G(u, v)$ or $d_{uv}$.

Let $V(G) = \{v_1, \ldots, v_n\}$. The distance matrix of $G$ is the $n \times n$ matrix $D(G) = (d_{v_i,v_j})$.

The eigenvalues of $D(G)$ are known as the distance eigenvalues of $G$. Label them as $\rho_1(G) \geq \cdots \geq \rho_n(G)$.

$\rho(G) = \rho_1(G)$: The distance spectral radius of $G$, i.e., the largest distance eigenvalue of $G$. 
Graham and Pollack established a relationship between the number of negative distance eigenvalues and the addressing problem in data communication systems, and they showed that the determinant of the distance matrix of a tree is a function of its order only.

Graham and Lovász computed the inverse of the distance matrix of a tree.


Generalizations on determinant and inverse of distance matrices of various classes of (di)graphs:


...
Recall that the inertia of the distance matrix of a tree is determined.


The inertia of the distance matrix of a unicyclic graph with an odd cycle is determined.

The characteristic polynomial of the distance matrix (called distance polynomial) of a graph was also investigated, and certain, and in some cases all, the coefficients of the distance characteristic polynomial are calculated.


Merris provided the first estimation of the distance spectrum of a tree using its Laplacian spectrum.


Ruzieh and Powers showed that the complete graph achieves the minimum and the path achieves the maximum distance spectral radius among connected graphs.

Koolen and Shpectorov proved that if the distance matrix of a distance-regular graph $G$ has exactly one positive eigenvalue then either $G$ is of diameter 2, or $G$ is an isometric subgraph of a halved cube.

J.H., Koolen, S.V. Shpectorov, Distance-regular graphs the distance matrix of which has only one positive eigenvalue, European J. Combin. 15 (1994) 269–275.

Conjectures

S. Fajtlowicz, Written on the wall, University of Houston, 1998.
Balaban, Ciubotariu and Medeleanu proposed the use of the distance spectral radius as a molecular descriptor. Gutman and Medeleanu used the distance spectral radius to infer the extent of branching and model boiling points of an alkane.


Lower and upper bounds for the distance spectral radius are found.


\ldots

Let $G$ be a non-trivial connected graph with $u \in V(G)$. For positive integers $k$ and $\ell$ with $k \geq \ell$, let $G_u(k, \ell)$ be the graph obtained from $G$ by attaching two pendant paths of length $k$ and $\ell$ respectively at $u$, and $G_u(k, 0)$ the graph obtained from $G$ by attaching a pendant path of length $k$ at $u$.

**Theorem**

Let $H$ be a non-trivial connected graph with $u \in V(H)$. If $k \geq \ell \geq 1$, then $\rho(H_u(k, \ell)) < \rho(H_u(k + 1, \ell - 1))$. 
Let $G$ be a connected graph with $u$ being a cut vertex. Suppose that $G_1$, $G_2$ and $G_3$ are subgraphs of $G$ such that $|V(G_i)| \geq 2$ for $1 \leq i \leq 3$, $V(G_i) \cap V(G_j) = \{u\}$ for $1 \leq i < j \leq 3$ and $\bigcup_{i=1}^{3} V(G_i) = V(G)$. For $v \in V(G_2) \setminus \{u\}$, let

$$G' = G - \{uw : w \in N_{G_3}(u)\} + \{vw : w \in N_{G_3}(u)\}.$$
For \( w \in V(G) \), define

\[
f(w) = \sum_{z \in V(G_1)} d_G(w, z) - \sum_{z \in V(G_2) \setminus \{u\}} d_G(w, z).
\]

**Theorem**

If \( f(w) \geq 0 \) for all \( w \in V(G_3) \setminus \{u\} \), and for some nonnegative number \( k \),

\[
\min_{w \in V(G_1)} f(w) \geq -k \quad \text{and} \quad \min_{w \in V(G_2) \setminus \{u\}} f(w) \geq k,
\]

then \( \rho(G') > \rho(G) \).
Theorem

If $|V(G_1)| \geq |V(G_2)| - 1$, $f(u) \geq 0$ and

$$\min_{w \in V(G_2) \setminus \{u\}} f(w) \geq \max \left\{ 0, - \min_{w \in V(G_1)} f(w) \right\},$$

then $\rho(G') > \rho(G)$. 
Let $G_1(s, t)$ be the graph shown below, where $G_1$ is a nontrivial connected graph, and $s, t \geq 1$.

Theorem

Let $G_1$ be a nontrivial connected graph. For $s \geq t \geq 2$, 

$$\rho(G_1(s + 1, t - 1)) > \rho(G_1(s, t)).$$
Among connected graphs on $n$ vertices, the complete graph achieves uniquely minimum distance spectral radius, the path achieves uniquely maximum distance spectral radius.

Among trees on $n$ vertices, the star achieves uniquely minimum distance spectral radius.


Theorem

Among trees with $n \geq 6$ vertices,
$D(n; 1, n - 3)$ achieves uniquely 2nd minimum distance spectral radius,
$S(n; 1, 1, n - 3)$ achieves uniquely 2nd maximum distance spectral radius;
$D(n; 2, n - 4)$ achieves uniquely 3rd minimum distance spectral radius,
$S(n; 1, 2, n - 4)$ achieves uniquely 3rd maximum distance spectral radius.
Theorem
Among non-caterpillar trees on $n \geq 7$ vertices, $B(n; n - 7, 1, 1, 1)$ is the unique graph with minimum distance spectral radius.

Theorem
Among non-starlike non-caterpillar trees on $n \geq 8$ vertices, $B(n; n - 8, 1, 1, 2)$ is the unique graph with minimum distance spectral radius.
Theorem

Among non-starlike trees on $n \geq 6$ vertices, $D_n$ is the unique graph with maximal distance spectral radius.

Theorem

Among non-caterpillar trees on $n \geq 7$ vertices, $S(n; 2, 2, n-5)$ is the unique graph with maximum distance spectral radius.
For $n \geq 8$, let $P = v_1 v_2 \cdots v_{n-3}$, let $P(n; 2, n-5)$ be the tree obtained from $P$ by attaching a pendant vertex to $v_2$ and a path $P_2 = v_n v_{n-1}$ at the terminal vertex to $v_{n-5}$.

**Theorem**

Among non-starlike non-caterpillar trees on $n \geq 8$ vertices, $P(n; 2, n-5)$ is the unique graph with maximal distance spectral radius.
Theorem

Among connected graphs with $n$ vertices and domination number $\gamma$, where $1 \leq \gamma \leq \lfloor \frac{n}{2} \rfloor$, $D \left( n, \left\lfloor \frac{n-3\gamma+2}{2} \right\rfloor, \left\lfloor \frac{n-3\gamma+2}{2} \right\rfloor \right)$ for $1 \leq \gamma < \lfloor \frac{n}{3} \rfloor$, $E \left( n, \left\lfloor \frac{3\gamma-n}{2} \right\rfloor, \left\lfloor \frac{3\gamma-n}{2} \right\rfloor \right)$ for $\lfloor \frac{n}{3} \rfloor < \gamma \leq \lfloor \frac{n}{2} \rfloor$ are the unique graphs with maximum distance spectral radius.
Theorem

Among trees with \( n \) vertices and \( 2k \) odd vertices, where \( 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \), \( C\left(n, \left\lfloor \frac{k-1}{2} \right\rfloor, \left\lceil \frac{k-1}{2} \right\rceil\right) \) is the unique tree with maximum distance spectral radius.
Watanabe et al. studied some spectral properties of the distance matrix of a uniform hypertree.

\( T_G(u) \): the transmission of \( u \) in \( G \), i.e.,

\[
T_G(u) = \sum_{v \in V(G)} d_G(u, v).
\]

\( T(G) \): the diagonal matrix of transmissions of \( G \).

\( Q(G) = T(G) + D(G) \): the distance signless Laplacian matrix of \( G \).

\( L(G) = T(G) - D(G) \): the distance Laplacian matrix of \( G \).

M. Aouchiche, P. Hansen, Two Laplacians for the distance matrix of a graph, Linear Algebra Appl. 439 (2013) 21–33.

$D_{\alpha}(G)$: the convex combinations of $T(G)$ and $D(G)$, defined as

$$D_{\alpha}(G) = \alpha T(G) + (1 - \alpha)D(G), \alpha \in [0, 1).$$

The eigenvalues of $D_{\alpha}(G)$ are called the distance $\alpha$-eigenvalues of $G$, and the largest distance $\alpha$-eigenvalue of $G$ is called the distance $\alpha$-spectral radius of $G$, written as $\mu_{\alpha}(G)$. 
A connected graph $G$ on $n$ vertices is distinguished vertex deleted regular (DVDR) if there is a vertex $v$ of degree $n - 1$ such that $G - v$ is regular.
A connected graph $G$ on $n$ vertices is **distinguished vertex deleted regular (DVDR)** if there is a vertex $v$ of degree $n - 1$ such that $G - v$ is regular.

Let $G$ be a connected graph and $u$ and $v$ be vertices such that $T_G(u) = T_{\text{min}}(G)$ and $T_G(v) = T_{\text{max}}(G)$. Let

\[ m_1 = \max\{T_G(w) - (1 - \alpha)d(u, w) : w \in V(G) \setminus \{u\}\}, \]

\[ m_2 = \min\{T_G(w) - (1 - \alpha)d(v, w) : w \in V(G) \setminus \{v\}\}, \]

and

\[ e(w) = \max\{d(w, z) : z \in V(G)\} \text{ for } w \in V(G). \]
Then
\[
m_2 + \alpha T_{\text{max}}(G) + \sqrt{(m_2 - \alpha T_{\text{max}}(G))^2 + 4(1 - \alpha)^2 T_{\text{max}}(G)} \leq \mu_{\alpha}(G) \leq m_1 + \alpha T_{\text{min}}(G) + \sqrt{(m_1 - \alpha T_{\text{min}}(G))^2 + 4(1 - \alpha)^2 e(u) T_{\text{min}}(G)}.
\]

The first equality holds if and only if \( G \) is a complete graph and the second equality holds if and only if \( G \) is a DVDR graph.

Let \( G \) be a connected graph of order \( n \geq 4 \), where \( G \not\cong P_n \). Then \( \mu_{\alpha}(G) \leq \mu_{\alpha}(B_{n,3}) < \mu_{\alpha}(P_n) \) with equality if and only if \( G \cong B_{n,3} \).


M. Aouchiche, P. Hansen, Two Laplacians for the distance matrix of a graph, Linear Algebra Appl. 439 (2013) 21–33.


谢谢 !!!