Some new signed Euler-Mahonian identities and polynomials

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Outline

1. Signed Euler-Mahonian Identities
2. Extensions to Coxeter group of type $B_n$
3. Extensions to Coxeter Groups of type $D_n$
4. Extensions to Complex Reflection Groups $G(r, 1, n)$
\( \mathcal{S}_n \) := the set of permutations of \( \{1, 2, \ldots, n\} \)

- \( \mathcal{S}_n = \langle s_1, s_2, \ldots, s_{n-1} \rangle \), where \( s_i = (i \ i + 1) \).
- \( \ell(\pi) := \) the minimal number of generators needed to represent \( \pi \)
  - \( 41253 = (3 \ 4)(2 \ 3)(4 \ 5)(1 \ 2) = s_3s_2s_4s_1 \)
  - \( \ell(41253) = 4 \)

- number of inversions: \( \text{inv}(\pi) := |\{(i, j) : i < j \text{ and } \pi_i > \pi_j\}| \).
  - \( \text{inv}(41253) = 3 + 0 + 0 + 1 = 4 \)
- \( \text{inv}(\pi) = \ell(\pi) \) for \( \pi \in \mathcal{S}_n \).
\( \mathcal{S}_n := \) the set of permutations of \( \{1, 2, \ldots, n\} \)

- \( \mathcal{S}_n = \langle s_1, s_2, \ldots, s_{n-1} \rangle \), where \( s_i = (i \ i + 1) \).
- \( \ell(\pi) := \) the minimal number of generators needed to represent \( \pi \)
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  - \( \text{inv}(41253) = 3 + 0 + 0 + 1 = 4 \)

- \( \text{inv}(\pi) = \ell(\pi) \) for \( \pi \in \mathcal{S}_n \).
The descent number \textbf{des} and major index \textbf{maj}

\[
\text{Des}(\pi) := \{i : \pi_i > \pi_{i+1}, i = 1, 2, \ldots, n - 1\}
\]

- descent of \(\pi\): \textbf{des}(\pi) := |\text{Des}(\pi)| \quad \triangleright \text{des}(41253) = 2
- major of \(\pi\): \textbf{maj}(\pi) := \sum_{i \in \text{Des}(\pi)} i \quad \triangleright \text{maj}(41253) = 1 + 4 = 5

\[\sum_{\pi \in S_n} q^{\text{maj}(\pi)} = \sum_{\pi \in S_n} q^{\text{inv}(\pi)}\]

- inv and \textbf{maj} are called \textit{Mahonian} statistics (equi-distributed with \(\ell\))
- \textbf{des} is called \textit{Eulerian} statistic

The descent number \( \text{des} \) and major index \( \text{maj} \)

\[ \text{Des}(\pi) := \{ i : \pi_i > \pi_{i+1}, i = 1, 2, \ldots, n - 1 \} \]

- descent of \( \pi \): \( \text{des}(\pi) := |\text{Des}(\pi)| \)
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- major of \( \pi \): \( \text{maj}(\pi) := \sum_{i \in \text{Des}(\pi)} i \)
  \[ \triangleright \text{maj}(41253) = 1 + 4 = 5 \]

**Theorem (MacMahon, 1913)**

\[
\sum_{\pi \in \mathfrak{S}_n} q^{\text{maj}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\text{inv}(\pi)}
\]

- \( \text{inv} \) and \( \text{maj} \) are called *Mahonian* statistics (equi-distributed with \( \ell \))
- \( \text{des} \) is called *Eulerian* statistic

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Signed Euler-Mahonian identities

Theorem (Désarménien-Foata, 1992)

\[
\sum_{\pi \in S_{2n}} (-1)^{\ell(\pi)} t^{\text{des}(\pi)} = (1 - t)^n \sum_{\pi \in S_n} t^{\text{des}(\pi)}
\]

\[
\sum_{\pi \in S_{2n+1}} (-1)^{\ell(\pi)} t^{\text{des}(\pi)} = (1 - t)^n \sum_{\pi \in S_{n+1}} t^{\text{des}(\pi)}
\]

Theorem (Wachs, 1992)

\[
\sum_{\pi \in S_{2n}} (-1)^{\ell(\pi)} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} = \prod_{i=1}^{n} (1 - tq^{2i-1}) \sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{2\text{maj}(\pi)}
\]

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Signed Euler-Mahonian identities

**Theorem (Désarménien-Foata, 1992)**

\[
\sum_{\pi \in S_{2n}} (-1)^{\ell(\pi)} t^{\text{des}(\pi)} = (1 - t)^n \sum_{\pi \in S_n} t^{\text{des}(\pi)} \\
\sum_{\pi \in S_{2n+1}} (-1)^{\ell(\pi)} t^{\text{des}(\pi)} = (1 - t)^n \sum_{\pi \in S_{n+1}} t^{\text{des}(\pi)}
\]

**Theorem (Wachs, 1992)**

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\]

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Extend to Coxeter groups

- In $\mathfrak{S}_n$: Coxeter group of type $A_{n-1}$
  - Generalized by $\langle s_1, s_2, \ldots, s_{n-1} \rangle$, where $s_i = (i \ i + 1)$
  - $\mathfrak{S}_n = \text{permutations of } \{1, 2, \ldots, n\}$
    - e.g. $\mathfrak{S}_2 = \{12, 21\}$
  - Signed Euler-Mahonian identity: $\sum_{\pi \in \mathfrak{S}_{2n}} (-1)^{\ell(\pi)} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} = \ldots$

- In $B_n$: Coxeter group of type $B_n$
  - Generalized by $\langle s_0, s_1, s_2, \ldots, s_{n-1} \rangle$, where $s_0 = (\overline{1} \ \overline{1})$
  - $B_n = \text{signed permutations of } \{1, 2, \ldots, n\}$
    - e.g. $B_2 = \{12, 1\overline{2}, \overline{1}2, \overline{1}\overline{2}, 21, 2\overline{1}, \overline{2}1, \overline{2}\overline{1}\}$
  - $\ell_B = \text{inv}(\pi) + \sum_{\pi_i < 0} |\pi_i|$
  - des??
  - maj??
Extend to Coxeter groups

- In $\mathfrak{S}_n$: Coxeter group of type $A_{n-1}$
  - Generalized by $\langle s_1, s_2, \ldots, s_{n-1} \rangle$, where $s_i = (i \, i + 1)$
  - $\mathfrak{S}_n = \text{permutations of } \{1, 2, \ldots, n\}$
    - e.g. $\mathfrak{S}_2 = \{12, 21\}$
  - Signed Euler-Mahonian identity: $\sum_{\pi \in \mathfrak{S}_{2n}} (-1)^{\ell(\pi)} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} = \ldots$

- In $\mathfrak{B}_n$: Coxeter group of type $B_n$
  - Generalized by $\langle s_0, s_1, s_2, \ldots, s_{n-1} \rangle$, where $s_0 = (\bar{1} \, 1)$
  - $\mathfrak{B}_n = \text{signed permutations of } \{1, 2, \ldots, n\}$
    - e.g. $\mathfrak{B}_2 = \{12, 1\bar{2}, \bar{1}2, \bar{1}\bar{2}, 21, 2\bar{1}, 2\bar{1}, \bar{2}1\}$
  - $\ell_B = \text{inv}(\pi) + \sum_{\pi_i < 0} |\pi_i|$
  - des??
  - maj??
Flag descent and major for $B_n$

$\Des_F(\pi) := \{ i : \pi_i > \pi_{i+1} \}$ w.r.t. $\bar{1} < \cdots < \bar{n} < 1 < \cdots < n$

- $\fdes(\pi) := 2 \cdot |\Des_F(\pi)| + \delta(\pi_i < 0)$
- $\fmaj(\pi) := 2 \cdot \sum_{i \in \Des_F(\pi)} \neg(\pi)$
  - $\Des_F(\bar{3}1\bar{6}2\bar{5}4) = \{2, 3\}$
  - $\fdes(\bar{3}1\bar{6}2\bar{5}4) = 2 \cdot 2 + 1 = 5$
  - $\fmaj(\bar{3}1\bar{6}2\bar{5}4) = 2 \cdot 5 + 4 = 14$

Theorem (Adin-Roichman, 2001)

\[
\sum_{\pi \in B_n} q^{\fmaj(\pi)} = \sum_{\pi \in B_n} q^{\ell_B(\pi)}
\]

Signed Euler-Mahonian identities for $\mathcal{B}_n$

**Theorem (Biagioli, 2006)**

\[
\sum_{\pi \in \mathcal{B}_{2n}} (-1)^{\text{inv}_B(\pi)} q^{\text{maj}(\pi)} = \prod_{i=1}^{n} \left(1 - q^{4i-2}\right) \sum_{\pi \in \mathcal{B}_n} q^{2\text{maj}(\pi)}
\]

**Theorem (–, preprints)**

\[
\sum_{\pi \in \mathcal{B}_{2n}} (-1)^{\text{inv}_B(\pi)} t^{\text{fdes}(\pi)} q^{\text{maj}(\pi)} = \prod_{i=1}^{n} \left(1 - t^2 q^{4i-2}\right) \sum_{\pi \in \mathcal{B}_n} t^{\text{fdes}(\pi)} q^{2\text{maj}(\pi)}
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Signed Euler-Mahonian identities for \( B_n \)

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\sum_{\pi \in B_{2n}} (-1)^{\text{inv}_B(\pi)} q^{\text{fmaj}(\pi)} = \prod_{i=1}^{n} \left( 1 - q^{4i-2} \right) \sum_{\pi \in B_n} q^{2\text{fmaj}(\pi)}
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\sum_{\pi \in B_{2n}} (-1)^{\text{inv}_B(\pi)} t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)} = \prod_{i=1}^{n} \left( 1 - t^2 q^{4i-2} \right) \sum_{\pi \in B_n} t^{\text{fdes}(\pi)} q^{2\text{fmaj}(\pi)}
\]

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Extend to 1-dim characters

- **1-dim character** of $G$ is a mapping $\chi : G \to \mathbb{C}$ being a **homomorphism**.

- **1-dim character** $\chi$ of Coxeter groups:
  - For $\mathfrak{S}_n$: $\chi(\pi) = 1, (-1)^{\ell(\pi)}$
  - For $\mathfrak{B}_n$: $\chi(\pi) = 1, (-1)^{\ell_B(\pi)}, (-1)^{\text{neg}(\pi)}, (-1)^{\text{inv}(|\pi|)}$

Signed Euler-Mahonian Polynomial

$$
\sum_{\pi \in G} \chi(\pi) t^{\text{stat}_1(\pi)} q^{\text{stat}_2(\pi)}
$$

- $\chi$: any 1-dim character of $G$
- $\text{stat}_1$: Eulerian statistic
- $\text{stat}_2$: Mahonian statistic
Extend to 1-dim characters

- **1-dim character** of $G$ is a mapping $\chi : G \to \mathbb{C}$ being a homomorphism.

- **1-dim character** $\chi$ of Coxeter groups:
  - For $S_n$: $\chi(\pi) = 1, (-1)^{\ell(\pi)}$
  - For $B_n$: $\chi(\pi) = 1, (-1)^{\ell_B(\pi)}, (-1)^{\text{neg}(\pi)}, (-1)^{\text{inv}(\pi)}$

**Signed Euler-Mahonians**

$$\sum_{\pi \in G} \chi(\pi) t^{\text{stat}_1(\pi)} q^{\text{stat}_2(\pi)}$$

- $\chi$: any 1-dim character of $G$
- $\text{stat}_1$: Eulerian statistic
- $\text{stat}_2$: Mahonian statistic
Main result 1: signed Euler-(Mahonian) identities for $\mathcal{B}_n$

**Theorem (Wachs, 1992)**

$$\sum_{\pi \in S_{2n}} (-1)^{\ell(\pi)} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} = \prod_{i=1}^{n} \left(1 - tq^{2i-1}\right) \sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{2\text{maj}(\pi)}$$

**Theorem (–, preprints)**

- $$\sum_{\pi \in \mathcal{B}_{2n}} (-1)^{\ell_B(\pi)} t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)} = \prod_{i=1}^{n} \left(1 - t^2 q^{4i-2}\right) \sum_{\pi \in \mathcal{B}_n} t^{\text{fdes}(\pi)} q^{2\text{fmaj}(\pi)}$$

- $$\sum_{\pi \in \mathcal{B}_{2n}} (-1)^{\text{inv}(\pi)} t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)} = \prod_{i=1}^{n} \left(1 - t^2 q^{4i-2}\right) \sum_{\pi \in \mathcal{B}_n} t^{\text{fdes}(\pi)} q^{2\text{fmaj}(\pi)}$$

- $$\sum_{\pi \in \mathcal{B}_n} (-1)^{\text{neg}(\pi)} t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)} = \sum_{\pi \in \mathcal{B}_n} t^{\text{fdes}(\pi)} (-q)^{\text{fmaj}(\pi)}$$
Main result 1: signed Euler-(Mahonian) identities for $\mathcal{B}_n$

Theorem (Désarménien-Foata, 1992)

\[
\sum_{\pi \in S_{2n+1}} (-1)^{\ell(\pi)} t^{\text{des}(\pi)} = (1 - t)^n \sum_{\pi \in S_{n+1}} t^{\text{des}(\pi)}
\]

Theorem (–, preprints)

\[
\sum_{\pi \in \mathcal{B}_{2n+1}} (-1)^{\text{inv}(|\pi|)} t^{\text{fdes}(\pi)} = (1 - t^2)^n \sum_{\pi \in \mathcal{B}_{n+1}} t^{\text{fdes}(\pi)}
\]

\[
\sum_{\pi \in \mathcal{B}_{2n+1}} (-1)^{\ell_B(\pi)} t^{\text{fdes}(\pi)} = (1 - t^2)^n (1 - t) \sum_{\pi \in \mathcal{B}_n} t^{2 \cdot \text{des}_B(\pi)}
\]

\[
\text{des}_B(\pi) := |\{i : \pi_i > \pi_{i+1}, 0 \leq i < n\}|, \text{ where } \pi_0 := 0
\]
Extensions to Coxeter Groups of type $D_n$
Main result 2: signed Euler-(Mahonian) identities for $\mathcal{D}_n$

$\mathcal{D}_n = \text{even-signed permutations of } \{1, 2, \ldots, n\}$

- Generalized by $\langle s'_0, s_1, s_2, \ldots, s_{n-1} \rangle$, where $s'_0 = (1 \ 2)$
  - $\mathcal{D}_2 = \{12, 1\bar{2}, 21, 2\bar{1}\}$

- $\mathcal{D}_n$ has two 1-dim characters: 1 and $(-1)^{\ell_D}$

- $d\text{des}(\pi) := f\text{des}(\pi_1 \pi_2 \cdots \pi_{n-1} | \pi_n |)$

- $d\text{maj}(\pi) := f\text{maj}(\pi_1 \pi_2 \cdots \pi_{n-1} | \pi_n |)$

**Theorem (–, preprints)**

\[
\sum_{\pi \in \mathcal{D}_{2n}} (-1)^{\ell_D} t^{d\text{des}(\pi)} q^{d\text{maj}(\pi)} = \prod_{i=1}^{n} (1 - t^2 q^{4i-2}) \sum_{\pi \in \mathcal{D}_n} t^{d\text{des}(\pi)} q^{2d\text{maj}(\pi)}
\]

\[
\sum_{\pi \in \mathcal{D}_{2n+1}} (-1)^{\ell_D(\pi)} t^{d\text{des}(\pi)} = (1 - t^2)^n \sum_{\pi \in \mathcal{D}_{n+1}} t^{d\text{des}(\pi)}
\]
Main result 2: signed Euler-(Mahonian) identities for $\mathcal{D}_n$

$\mathcal{D}_n = \text{even-signed permutations of } \{1, 2, \ldots, n\}$

- Generalized by $\langle s'_0, s_1, s_2, \ldots, s_{n-1} \rangle$, where $s'_0 = (1 \ 2)$
  - $\mathcal{D}_2 = \{12, 1\bar{2}, 21, 2\bar{1}\}$

- $\mathcal{D}_n$ has two 1-dim characters: 1 and $(-1)^{\ell_D}$
- $\text{ddes}(\pi) := \text{fdes}(\pi_1 \pi_2 \cdots \pi_{n-1} | \pi_n |)$
- $\text{dmaj}(\pi) := \text{fmaj}(\pi_1 \pi_2 \cdots \pi_{n-1} | \pi_n |)$

**Theorem (–, preprints)**

$$
\sum_{\pi \in \mathcal{D}_{2n}} (-1)^{\ell_D} t^{\text{ddes}(\pi)} q^{\text{dmaj}(\pi)} = \prod_{i=1}^{n} (1 - t^2 q^{4i-2}) \sum_{\pi \in \mathcal{D}_n} t^{\text{ddes}(\pi)} q^{2\text{dmaj}(\pi)}
$$

$$
\sum_{\pi \in \mathcal{D}_{2n+1}} (-1)^{\ell_D(\pi)} t^{\text{ddes}(\pi)} = (1 - t^2)^n \sum_{\pi \in \mathcal{D}_{n+1}} t^{\text{ddes}(\pi)}
$$
Proof Sketch for even case: Step 1

For \( \pi \in \mathcal{D}_{2n} \) let \( i \) be the smallest integer such that \( 2i - 1 \) and \( 2i \)

1. have opposite signs,
2. are not in adjacent positions, or
3. are both at the last two positions with negative signs.

Let \( \eta(\pi) \) be the even-signed permutation obtained from \( \pi \) by swapping the two letters \( 2i - 1 \) and \( 2i \).

\[ \eta(213564) = 214563, \quad \eta(213564) = 123564, \quad \eta(213456) = 213465 \]

Fixed points \( F_{2n} : \) letters \( 2i - 1 \) and \( 2i \) are adjacent and having the same sign, and both \( \pi_{2n-1} \) and \( \pi_{2n} \) are positive

\[ \ell_D(\pi) = \ell_D(\pi') \pm 1 \]

\[ \text{Des}_F(\pi_1\pi_2\cdots\pi_{2n-1}|\pi_{2n}|) = \text{Des}_F(\pi'_1\pi'_2\cdots\pi'_{2n-1}|\pi'_{2n}|) \]
Proof Sketch for even case: Step 1

For \( \pi \in \mathcal{D}_{2n} \) let \( i \) be the smallest integer such that \( 2i - 1 \) and \( 2i \)

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e.g. \( \eta(213564) = 214563, \quad \eta(213564) = 123564, \quad \eta(213456) = 213465 \)

Fixed points \( \mathcal{F}_{2n} \): letters \( 2i - 1 \) and \( 2i \) are adjacent and having the same sign, and both \( \pi_{2n-1} \) and \( \pi_{2n} \) are positive

- \( \ell_D(\pi) = \ell_D(\pi') \pm 1 \)
- \( \text{Des}_F(\pi_1 \pi_2 \cdots \pi_{2n-1} | \pi_{2n}) = \text{Des}_F(\pi'_1 \pi'_2 \cdots \pi'_{2n-1} | \pi'_{2n}) \)
Proof Sketch for even case: Step 2

Define a bijective correspondence \( \phi: \mathcal{F}_{2n} \to \hat{\mathcal{D}}_n \) as

1. Each pair of adjacent entries of type \( \pm(2j-1), \pm 2j \) in \( \mathcal{F}_{2n} \) is replaced by \( \pm j \);
2. Each pair of adjacent entries of type \( \pm 2j, \pm (2j-1) \) in \( \mathcal{F}_{2n} \) is replaced by \( \pm \hat{j} \);
3. After the two steps, if the number of negatives of the resulting permutation is odd, then change the sign of the last entry from positive to negative.

\[ \phi(\pi) = \phi(21\bar{5}\bar{6}8743) = \hat{1}\bar{3}\hat{4}\bar{2} = \pi' \]

- \((-1)^{\ell_D(\pi)} = (-1)^{|P(\pi')|}\)
- \(ddes(\pi) = ddes(\pi') + 2|P(\pi')|\)
- \(dmaj(\pi) = 2dmaj(\pi') + \sum_{i \in P(\pi')} (4i - 2)\)
Proof Sketch for even case: Step 2

Define a bijective correspondence $\phi : \mathcal{F}_{2n} \to \hat{\mathcal{D}}_n$ as

1. Each pair of adjacent entries of type $\pm(2j - 1), \pm 2j$ in $\mathcal{F}_{2n}$ is replaced by $\pm j$;
2. Each pair of adjacent entries of type $\pm 2j, \pm(2j - 1)$ in $\mathcal{F}_{2n}$ is replaced by $\pm \hat{j}$;
3. After the two steps, if the number of negatives of the resulting permutation is odd, then change the sign of the last entry from positive to negative.

E.g. $\phi(\pi) = \phi(21\bar{5}\bar{6}8743) = \hat{1}\bar{3}\hat{4}\hat{2} = \pi'$

- $(-1)^{\ell_D(\pi)} = (-1)^{|P(\pi')|}$
- $\text{ddes}(\pi) = \text{ddes}(\pi') + 2|P(\pi')|$
- $\text{dmaj}(\pi) = 2\text{dmaj}(\pi') + \sum_{i \in P(\pi')} (4i - 2)$
Proof Sketch for even case: Step 3

\[ \sum_{\pi \in \mathcal{D}_{2n}} (-1)^{\ell_D} t^{\text{ddes}(\pi)} q^{\text{dmaj}(\pi)} \]

(Step 1) \[ = \sum_{\pi \in \mathcal{F}_{2n}} (-1)^{\ell_D} t^{\text{ddes}(\pi)} q^{\text{dmaj}(\pi)} \]

(Step 2) \[ = \sum_{\pi' \in \hat{\mathcal{D}}_n} \left( (-1)^{|\mathcal{P}(\pi')|} t^{2|\mathcal{P}(\pi')|} q^{\sum_{i \in \mathcal{P}(\pi')} (4i-2)} \right) t^{\text{ddes}(\pi')} q^{2\text{dmaj}(\pi')} \]

\[ = \sum_{\pi' \in \mathcal{D}_n} \left( \sum_{A \subseteq \{1,2,\ldots,n\}} (-1)^{|A|} t^{|A|} q^{\sum_{i \in A} (4i-2)} \right) t^{\text{ddes}(\pi')} q^{2\text{dmaj}(\pi')} \]

\[ = \prod_{i=1}^{n} (1 - t^2 q^{4i-2}) \sum_{\pi' \in \mathcal{D}_n} t^{\text{ddes}(\pi')} q^{2\text{dmaj}(\pi')} . \]
Proof Sketch for odd case: Step 1

For $\pi \in D_{2n+1}$ let $i$ be the smallest integer such that $2i - 1$ and $2i$

1. are not in adjacent positions,
2. have opposite signs, and are not both at the last two positions, or
3. are both at the last two positions and $\pi_{2n} < 0$

Let $\iota(\pi)$ be the even-signed permutation obtained from $\pi$ by swapping the two letters $2i - 1$ and $2i$.

e.g. $\iota(587129634) = 687129534$, $\iota(587129634) = 587129643$,
$\iota(587129634) = 587129643$

Fixed points $F_{2n+1}$:

- Letters $2i - 1$ and $2i$ are adjacent.
- Letters $2i - 1$ and $2i$ have the same sign if both of them are not at the last two positions.
- If $2i - 1$ and $2i$ appear at the last two positions for some $i$, then $\pi_{2n} > 0$. 
Proof Sketch for odd case: Step 1

For $\pi \in D_{2n+1}$ let $i$ be the smallest integer such that $2i - 1$ and $2i$

1. are not in adjacent positions,
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Fixed points $F_{2n+1}$:

- Letters $2i - 1$ and $2i$ are adjacent.
- Letters $2i - 1$ and $2i$ have the same sign if both of them are not at the last two positions.
- If $2i - 1$ and $2i$ appear at the last two positions for some $i$, then $\pi_{2n} > 0$. 
Proof Sketch for odd case: Step 2

Define a bijective correspondence $\phi : \mathcal{F}_{2n+1} \rightarrow \hat{\mathcal{D}}_{n+1}$ as

1. Each pair of adjacent entries of type $\pm(2j - 1), \pm 2j$ in $\mathcal{F}_{2n+1}$ but not at the last two positions is replaced by $\pm j$;

2. Each pair of adjacent entries of type $\pm 2j, \pm (2j - 1)$ in $\mathcal{F}_{2n+1}$ but not at the last two positions is replaced by $\pm \hat{j}$;

3. The pair of entries of type $(2j - 1), \pm 2j$ at the last two positions in $\mathcal{F}_{2n+1}$ is replaced by $\pm j$;

4. The pair of entries of type $2j, \pm (2j - 1)$ at the last two positions in $\mathcal{F}_{2n+1}$ is replaced by $\pm \hat{j}$;

5. The entry $\pm (2n + 1)$ in $\mathcal{F}_{2n+1}$ is replaced by $\pm (n + 1)$;

6. After the above steps, if the number of negatives of the resulting permutation is odd, then change the sign of the last entry.

e.g. $\phi(215\bar{6}\bar{8}7\bar{9}4\bar{3}) = 1\bar{3}\bar{4}5\bar{2}$ \quad $\phi(21\bar{5}\bar{6}\bar{8}7\bar{4}\bar{3}9) = 1\bar{3}\bar{4}\bar{2}\bar{5}$
Proof Sketch for odd case: Step 3

- \((-1)^{\ell_D(\pi)} = (-1)|P(\pi')| = (-1)|L(\pi')|\)
- \(d\text{des}(\pi) = d\text{des}(\pi') + 2|L(\pi')|\)

\[
\sum_{\pi \in \mathcal{F}_{2n+1}} (-1)^{\ell_D} t^{d\text{des}(\pi)} = \sum_{\pi' \in \hat{\mathcal{D}}_{n+1}} (-1)^{|L(\pi')|} t^{d\text{des}(\pi')} + 2|L(\pi')|
\]

\[
= \sum_{\pi' \in \mathcal{D}_{n+1}} \left( \sum_{A \in [n]} (-1)^{|A|} t^2|A| \right) t^{d\text{des}(\pi')} = (1 - t^2)^n \sum_{\pi' \in \mathcal{D}_{n+1}} t^{d\text{des}(\pi')}
\]
Extensions to Complex Reflection Groups $G(r, 1, n)$

$\mathcal{S}_n \xrightarrow{\ell_A} \theta_A$ des maj

$\mathcal{B}_n \xrightarrow{\ell_B} \theta_B$ fdes fmaj

$\mathcal{D}_n \xrightarrow{\ell_D} \theta_D$ ddes dmaj

Coxeter groups

Wreath Product

$\triangleright G(r, 1, n) \xleftarrow{\ell} fdes fmaj$
G(r, 1, n) and length function ℓ

\[ G(r, 1, n) := \mathbb{Z}r \wr S_n: \text{ Wreath product} \]
\[ = \text{colored permutation group on } \{1, 2, \ldots, n\} \text{ with } r \text{ colors} \]

\text{e.g.} \quad 1 \text{ color, 6 letters: } 5 6 3 1 4 2
\quad 4 \text{ colors, 6 letters: } \begin{array}{c}
5 \bar{6} \bar{3} \bar{1} 4 2 \\
\end{array}

\begin{itemize}
\item G(1, 1, n) = S_n; \quad G(2, 1, n) = B_n.
\item G(r, 1, n) = \langle s_0, s_1, \ldots, s_{n-1} \rangle \text{ where } s_0 := \text{add one more bar on the first letter.}
\item ℓ(\pi) := \text{the minimal number of generators needed to represent } \pi
\end{itemize}
$G(r, 1, n)$ and length function $\ell$

$G(r, 1, n) := \mathbb{Z}r \wr \mathfrak{S}_n$: Wreath product

= colored permutation group on \{1, 2, \ldots, n\} with $r$ colors

\begin{itemize}
  \item e.g. 1 color, 6 letters: 5 6 3 1 4 2
  \item 4 colors, 6 letters: $\overline{5} \overline{6} \overline{3} \overline{1} \overline{4} \overline{2}$
\end{itemize}

\begin{itemize}
  \item $G(1, 1, n) = \mathfrak{S}_n$; \quad $G(2, 1, n) = \mathcal{B}_n$.
  \item $G(r, 1, n) = \langle s_0, s_1, \ldots, s_{n-1} \rangle$ where $s_0 := \text{add one more bar on the first letter}$.
  \item $\ell(\pi) := \text{the minimal number of generators needed to represent } \pi$
\end{itemize}
Represent $\chi$ in terms of $\ell$

**Theorem (–, arXiv)**

$G(r, 1, n)$ has $2r$ 1-dim characters

$$
\chi_{a,b}(\pi) = (-1)^{a(\ell(\pi) - \sum z_i)} \omega^b \sum z_i,
$$

where $\omega = e^{2\pi i / r}$, $a = 0, 1$ and $b = 0, 1, \ldots, r - 1$.

For $r = 2$, $\ell = \ell_B$, $\omega = -1$ and the four 1-dim characters are

- $\chi_{0,0} = (-1)^0(-1)^0 = 1$,
- $\chi_{0,1} = (-1)^0(-1)\sum z_i = (-1)^{\text{neg}(\pi)}$,
- $\chi_{1,0} = (-1)^{\ell(\pi) - \sum z_i}(-1)^0 = \cdots = (-1)^{\text{inv}(|\pi|)}$,
- $\chi_{1,1} = (-1)^{\ell(\pi) - \sum z_i}(-1)^{\sum z_i} = (-1)^{\ell_B}$.
Flag descent and major $G(r, 1, n)$

$$\text{Des}_F(\pi) := \{ i : \pi_i > \pi_{i+1} \} \text{ w.r.t.}$$

$$1^{[r-1]} < \cdots < n^{[r-1]} < \cdots < 1 \prec \cdots \prec n \prec 1 \prec \cdots \prec n$$

- $f\text{des}(\pi) := r \cdot |\text{Des}_F(\pi)| + z_1$
- $f\text{maj}(\pi) := r \cdot \sum_{i \in \text{Des}_F(\pi)} + \text{col}(\pi)$

- $\text{Des}_F(\bar{45\bar{1}\bar{3}\bar{2}\bar{6}}) = \{2, 4, 5\}$
- $f\text{des}(\bar{45\bar{1}\bar{3}\bar{2}\bar{6}}) = 3 \cdot 3 + 2 = 9$
- $f\text{maj}(\bar{45\bar{1}\bar{3}\bar{2}\bar{6}}) = 3 \cdot 11 + 8 = 41$

Main result 3

Theorem (–, preprints)

For \( b = 0, 1, \ldots, r - 1 \), we have

\[
\sum_{\pi \in G(r,1,2n)} \chi_{1,b}(\pi) t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)} = \prod_{i=1}^{n} \left( 1 - t^r q^{r(2i-1)} \right) \sum_{\pi \in G(r,1,n)} t^{\text{fdes}(\pi)} (\omega b q)^{2\text{fmaj}(\pi)},
\]

and

\[
\sum_{\pi \in G(r,1,n)} \chi_{0,b}(\pi) t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)} = \sum_{\pi \in G(r,1,n)} t^{\text{fdes}(\pi)} (\omega b q)^{\text{fmaj}(\pi)}.
\]

The case \( G(r,1,2n+1) \) with \( \chi_{1,b} \) for any \( b \) is missing!
Main result 3

Theorem (–, preprints)

For \( b = 0, 1, \cdots, r - 1 \), we have

\[
\sum_{\pi \in G(r,1,2n)} \chi_{1,b}(\pi) t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)}
\]

\[
= \prod_{i=1}^{n} \left(1 - t^r q^{r(2i-1)}\right) \sum_{\pi \in G(r,1,n)} t^{\text{fdes}(\pi)} (\omega^b q)^{2\text{fmaj}(\pi)},
\]

and

\[
\sum_{\pi \in G(r,1,n)} \chi_{0,b}(\pi) t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)} = \sum_{\pi \in G(r,1,n)} t^{\text{fdes}(\pi)} (\omega^b q)^{\text{fmaj}(\pi)}.
\]

The case \( G(r, 1, 2n + 1) \) with \( \chi_{1,b} \) for any \( b \) is missing!
Concluding Remarks
Signed Euler-Mahonian identities

Generalize the following two identities.

**Theorem (Wachs, 1992)**

\[
\sum_{\pi \in \mathcal{S}_{2n}} (-1)^{\text{inv}(\pi)} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} = \prod_{i=1}^{n} (1 - t q^{2i-1}) \sum_{\pi \in \mathcal{S}_n} t^{\text{des}(\pi)} q^{2\text{maj}(\pi)}
\]

**Theorem (Désarménien-Foata, 1992)**

\[
\sum_{\pi \in \mathcal{S}_{2n+1}} (-1)^{\text{inv}(\pi)} t^{\text{des}(\pi)} = (1 - t)^n \sum_{\pi \in \mathcal{S}_{n+1}} t^{\text{des}(\pi)}
\]
Signed Euler-Mahonian identities

**Theorem (Wachs, 1992)**

\[
\sum_{\pi \in \mathcal{S}_{2n}} (-1)^{\text{inv}(\pi)} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} = \prod_{i=1}^{n} (1 - t q^{2i} - 1) \sum_{\pi \in \mathcal{S}_{n}} t^{\text{des}(\pi)} q^{2\text{maj}(\pi)}
\]

\[
\sum_{\pi \in W} \chi(\pi) t^{\text{stat}_1} q^{\text{stat}_2} = \ldots
\]

- \( W = \mathcal{B}_{2n} \): \((\text{stat}_1, \text{stat}_2) = (\text{fdes}, \text{fmaj})\)
- \( W = \mathcal{D}_{2n} \): \((\text{stat}_1, \text{stat}_2) = (\text{ddes}, \text{dmaj})\)
- \( W = G(r, 1, 2n) \): \((\text{stat}_1, \text{stat}_2) = (\text{fdes}, \text{fmaj})\)
Signed Eulerian identities

**Theorem (Désarménien-Foata, 1992)**

\[
\sum_{\pi \in S_{2n+1}} (-1)^{\text{inv}(\pi)} t^{\text{des}(\pi)} = (1 - t)^n \sum_{\pi \in S_{n+1}} t^{\text{des}(\pi)}
\]

\[
\sum_{\pi \in W} \chi(\pi) t^{\text{stat}_1} q^{\text{stat}_2} = \ldots
\]

- \( W = B_{2n+1} \): \((\text{stat}_1, \text{stat}_2) = (\text{fdes}, \text{fmaj})\)
- \( W = D_{2n+1} \): \((\text{stat}_1, \text{stat}_2) = (\text{ddes}, \text{dmaj})\)
- \( W = G(r, 1, 2n + 1) \): ???
More Future works

Let $G(r, p, n)$ denote the *complex reflection group* with parameters $r, p, n$, where $p \mid r$.

- $G(1, 1, n) = S_n$, the Coxeter group of type $A_{n-1}$
- $G(2, 1, n) = B_n$, the Coxeter group of type $B_n$
- $G(2, 2, n) = D_n$, the Coxeter group of type $D_n$
- $G(r, 1, n) = C_r \wr S_n$, the Wreath product of $C_r$ with $S_n$

**Question.**

Is it possible to extend to $G(r, p, n)$?
Thank you for your attention