Weakly symmetric hexavalent graphs of order $9p$

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Outline

- Definitions
- Motivation
- The arc-transitive case
- Classification for arc-transitive case
- The half-arc-transitive case
- Classification for half-arc-transitive case
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Definitions

Let $X$ be a regular graph and $\text{Aut}(X)$ the full automorphism group.

Different types of transitivity

- **vertex-transitive**: $\text{Aut}(X)$ is transitive on vertices.
- **edge-transitive**: $\text{Aut}(X)$ is transitive on edges.
- **arc-transitive**: $\text{Aut}(X)$ is transitive on arcs.
- **half-arc-transitive**: $\text{Aut}(X)$ is transitive on vertices and edges but not on arcs.
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Let $X$ be a weakly symmetric graph, and $p, q$ two distinct primes.

- $|V(X)| = p$: by Chao in 1971, $X$ must be arc-transitive.

- $|V(X)| = 2p$: by Cheng and Oxley in 1987, $X$ must be arc-transitive.

- $|V(X)| = pq$: by Alspach, Praeger, Wang and Xu in 1994, $X$ can be arc-transitive or half-arc-transitive.

- $|V(X)| = 2p^2$: by Zhou and Zhang in 2018, $X$ must be arc-transitive.
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Motivation

Arc-transitive graphs

- **Characterization and classification on highly arc-transitive graphs**: Praeger, Li, Fang and Lu, et al.
- Such graphs with **certain primitive action**: Praeger, Li, Fang and Lu, et al.
- **Prime valency**: by using the structure of vertex stabilizers, and covering and lifting technique, for example, Feng, Marušič and Zhou, et al.
- **Four valency**: by Fang, Feng, Li, Lu and Zhou, et al.
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Half-arc-transitive graphs

- $|V(X)| = p, 2p$ or $2p^2$: All are arc-transitive.
- $|V(X)| = 4p$: by Kutnar, Marušič, et al. All are metacirculants.
- $|V(X)| = pq$: by Alspach, Xu, Wang and Dobson. All are metacirculants.
- Tetravalent case: by Conder, Marušič, Feng, Xu, Zhou, et al.
- $|V(X)| = p^3$: by Feng and Wang, an infinite family of non-metacirculants.

$NH$-number: stands for non-half-arc-transitive, defined by Zhou in 2018.
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The arc-transitive case

**Theorem 1.1, DM, 2017**

Let $p$ be a prime. Then any connected hexavalent arc-transitive graph of order $9p$ is isomorphic to one of the following graphs.

<table>
<thead>
<tr>
<th>$p$</th>
<th>s-transitive</th>
<th>$\text{Aut}(X)$</th>
<th>Num.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1-transitive</td>
<td>$(S_3 \times \mathbb{Z}<em>3).D</em>{12}$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1-transitive</td>
<td>$\text{Aut}(X)<em>v=D</em>{12}$, $S_4 \times \mathbb{Z}_2$, $D_8 \times S_3$</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>1-transitive</td>
<td>$\mathbb{Z}_3.S_6$</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1-transitive</td>
<td>$G_2(2)$</td>
<td>2</td>
</tr>
<tr>
<td>$p$</td>
<td>1-transitive</td>
<td>$S_3 \wr D_{6p}$</td>
<td>1</td>
</tr>
<tr>
<td>$p \geq 3$</td>
<td>1-transitive</td>
<td>$S_3 \wr D_{2p}$</td>
<td>1</td>
</tr>
<tr>
<td>$p \equiv 1 (\text{mod } 6)$</td>
<td>1-regular</td>
<td>$\mathbb{Z}_{9p} \rtimes \mathbb{Z}_6$</td>
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<tr>
<td>$p \equiv 1 (\text{mod } 6)$</td>
<td>1-regular</td>
<td>$(\mathbb{Z}_3^2 \times \mathbb{Z}_p) \rtimes \mathbb{Z}_6$</td>
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</table>
Classification for arc-transitive case

**Ideas of proof**

Let $X$ be such graph, $A = \text{Aut}(X)$ and $N$ a minimal normal subgroup of $A$.

- $X$ cannot be a normal Cayley graph on a non-abelian group.
  
  By using the automorphisms of non-abelian group of order $9p$.

- $X$ cannot be 2-arc-transitive.
  
  By using the structure of vertex stabilizers of 2-arc-transitive hexavalent graphs, quotient graphs relative to the orbits of a minimal normal subgroup of $A$, and the $K_3$- and $K_4$-simple groups.

- $N$ has two cases: $N_v \neq 1$ or $N_v = 1$.
  
  For $N_v \neq 1$, $X \cong C_{3p}[3K_1]$ or $C(3, p, 2)$.
  
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The half-arc-transitive case

**Theorem 1.1, DM, 2019**

Let $p$ be a prime and $X$ a connected hexavalent half-arc-transitive graph of order $9p$. Then $X$, the automorphism group $\text{Aut}(X)$ and the vertex stabilizer $\text{Aut}(X)_v$ for a vertex $v \in V(X)$ are described in the following table:

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<td>$9\mid (p - 1)$</td>
<td>$\mathbb{Z}_3 \rtimes \mathbb{Z}_9$</td>
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Classification for half-arc-transitive case

Ideas of proof

- Every minimal normal subgroup of $A$ is solvable.
  By using that $A_v$ is a $\{2, 3\}$-group and $K_3$-simple group.

- Every normal abelian 3-subgroup $M$ of $A$ is isomorphic to $\mathbb{Z}_3$.
  By using quotient graphs relative to the orbits of $M$. If $M \not\cong \mathbb{Z}_3$, then $X$ is arc-transitive.

- $A \cong \mathbb{Z}_3 \times (\mathbb{Z}_p \rtimes \mathbb{Z}_9)$ with $9 \mid (p - 1)$ or $\mathbb{Z}_p \rtimes \mathbb{Z}_{27}$ with $27 \mid (p - 1)$.
  By using the edge-transitive graphs of order $3p$ or 9.

- Classification.
  Constructing coset graph by the full automorphism group $A$. By using the $G\text{I}$-property and calculating the orbits of $A$ acting on the corresponding right cosets.
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- $A \cong \mathbb{Z}_3 \times (\mathbb{Z}_p \rtimes \mathbb{Z}_9)$ with $9 \mid (p - 1)$ or $\mathbb{Z}_p \rtimes \mathbb{Z}_{27}$ with $27 \mid (p - 1)$.
  By using the edge-transitive graphs of order $3p$ or 9.

Classification.

Constructing coset graph by the full automorphism group $A$. By using the GI-property and calculating the orbits of $A$ acting on the corresponding right cosets.
Ideas of proof

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Classification for half-arc-transitive case

Ideas of proof

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Classification.

Constructing coset graph by the full automorphism group $A$. By using the GI-property and calculating the orbits of $A$ acting on the corresponding right cosets.
Examples: arc-transitive case

Example (Definition 2.1, Praeger and Xu, European J. Combin., 1989)

Let $p \geq 3$. Then define the graph $C(3, p, 2) = (V, E)$ as follows:

$V(X) = \mathbb{Z}_p \times \mathbb{Z}_3^2$, \quad $E = \{((i, x, y), (i + 1, y, z))\}$

where $\mathbb{Z}_p$ and $\mathbb{Z}_3$ are additive groups of order $p$ and 3, $i \in \mathbb{Z}_p$ and $x, y, z \in \mathbb{Z}_3$. Then $C(3, p, 2)$ is a connected hexavalent symmetric graphs of order $9p$ and $\text{Aut}(C(3, p, 2)) = S_3 \text{ wr } D_{2p}$.

Remark. It is easy to check that $C(3, 3p, 1) \cong C_{3p}[3K_1]$. Clearly, $C(3, p, 2)$ is not isomorphic to $C_{3p}[3K_1]$ because

$\text{Aut}(C(3, p, 2)) \neq \text{Aut}(C_{3p}[3K_1])$.

$\text{Aut}(C(3, p, 2))$ has a minimal normal subgroup isomorphic to $\mathbb{Z}_3^p$, which is not semiregular on $V(C(3, p, 2))$. 
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$\text{Aut}(C(3, p, 2))$ has a minimal normal subgroup isomorphic to $\mathbb{Z}_3^p$, which is not semiregular on $V(C(3, p, 2))$. 
Let $p$ be a prime and $r$ an element of order 9 in $\mathbb{Z}_p^*$. Then $9 \mid (p - 1)$. Suppose that $G(3p \times 9) = \langle a, b, c \mid a^p = b^9 = c^3 = [a, c] = [b, c] = 1, b^{-1}ab = a' \rangle \cong \mathbb{Z}_3 \times (\mathbb{Z}_p \rtimes \mathbb{Z}_9)$ with $r \neq 1$.

Construction 4.3, DM, 2019

Take $H = \langle b^3c \rangle \leq G(3p \times 9)$ and $g = ab$. Then $H \cong \mathbb{Z}_3$. Define the following coset graph:

$$\mathcal{HC}_{3p \times 9}(9p) = \text{Cos}(G(3p \times 9), H, H\{g, g^{-1}\}H).$$

The coset graph $\mathcal{HC}_{3p \times 9}(9p)$ is a connected hexavalent half-arc-transitive graph of order $9p$, and

$$\text{Aut}(\mathcal{HC}_{3p \times 9}(9p)) \cong G(3p \times 9).$$
Let $p$ be a prime and $r$ an element of order 9 in $\mathbb{Z}_p^*$. Then $9 \mid (p - 1)$. Suppose that $G(3p \times 9) = \langle a, b, c \mid a^p = b^9 = c^3 = [a, c] = [b, c] = 1, b^{-1}ab = a^r \rangle \cong \mathbb{Z}_3 \times (\mathbb{Z}_p \rtimes \mathbb{Z}_9)$ with $r \neq 1$.

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Take $H = \langle b^3c \rangle \leq G(3p \times 9)$ and $g = ab$. Then $H \cong \mathbb{Z}_3$. Define the following coset graph:

$$\mathcal{H}C_{3p \times 9}(9p) = \text{Cos}(G(3p \times 9), H, H\{g, g^{-1}\}H).$$

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$$\text{Aut}(\mathcal{H}C_{3p \times 9}(9p)) \cong G(3p \times 9).$$
Examples: half-arc-transitive case

Let $p$ be a prime and $s$ an element of order 27 in $\mathbb{Z}_p^*$. Then $27 \mid (p - 1)$. Suppose that $G(p \times 27) = \langle x, y \mid x^p = y^{27} = 1, y^{-1}xy = x^s \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{27}$ with $s \neq 1$.

**Construction 4.5, DM, 2019**

Take $K = \langle y^9 \rangle \leq G(p \times 27)$. Then $K \cong \mathbb{Z}_3$. Set $g_1 = xy$, $g_2 = xy^2$, $g_3 = xy^4$. Define the following coset graphs:

- $HC_{p \times 27}(9p, 1) = \text{Cos}(G(p \times 27), K, K\{g_1, g_1^{-1}\}K)$;
- $HC_{p \times 27}(9p, 2) = \text{Cos}(G(p \times 27), K, K\{g_2, g_2^{-1}\}K)$;
- $HC_{p \times 27}(9p, 3) = \text{Cos}(G(p \times 27), K, K\{g_3, g_3^{-1}\}K)$.

The coset graphs $HC_{p \times 27}(9p, i)$ are connected hexavalent half-arc-transitive graphs of order $9p$, and for $i = 1, 2, 3$

$$\text{Aut}(HC_{p \times 27}(9p, i)) \cong G(p \times 27).$$
Let $p$ be a prime and $s$ an element of order 27 in $\mathbb{Z}_p^*$. Then $27 \mid (p - 1)$.

Suppose that $G(p \times 27) = \langle x, y \mid x^p = y^{27} = 1, y^{-1}xy = x^s \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{27}$ with $s \neq 1$.

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Take $K = \langle y^9 \rangle \leq G(p \times 27)$. Then $K \cong \mathbb{Z}_3$. Set $g_1 = xy$, $g_2 = xy^2$, $g_3 = xy^4$. Define the following coset graphs:

$$
\mathcal{H}C_{p \times 27}(9p, 1) = \text{Cos}(G(p \times 27), K, K\{g_1, g_1^{-1}\}K);
$$

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$$
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$$
Thank you!