Disjoint cycles in digraphs

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1. Notation

- A graph \( G = (V, E) \): \( V \) vertex set, \( E \) edge set.
  
  Let \( n \) be the order of \( G \) for simplicity, i.e. \( n = |V| \).

- \( \delta(G) \): the minimum degree of \( G \).

- A digraph \( D = (V, A) \): \( V \) vertex set, \( A \) arc set.

- \( \delta^+(D) \): the minimum out-degree,

- \( \delta^-(D) \): the minimum in-degree of \( D \).

- The semi-degree of \( D \) is \( \delta^0(D) = \min\{\delta^+(D), \delta^-(D)\} \).

- In a digraph: a cycle is always **directed**.
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- In a digraph: a cycle is always directed.
- A **tournament** is a digraph $T$ such that for any two distinct vertices $x$ and $y$, exactly one of the ordered pairs $(x, y)$ and $(y, x)$ is an arc of $T$.

- A set of subgraphs of $G$ or $D$ is said to be **vertex-disjoint** (briefly **disjoint**) if no two of them have any common vertex in $G$ or $D$.

- A **2-factor** of a graph $G$ is a spanning subgraph of $G$ such that each component is a cycle. A **hamiltonian cycle** is a 2-factor with exactly one component.

- A **cycle factor** of a digraph $D$ is a spanning subgraph of $D$ such that each component is a cycle in $D$. 
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2. Introduction and Main Theorem

For a graph $G$, if $e(G) \geq n$, or $\delta(G) \geq 2$, then $G$ contains a cycle.

How about two disjoint cycles?

P. Erdős, L. Pósa, 1962

For every graph $G$, if $n \geq 6$ and $e(G) \geq 3n - 6$, then $G$ has 2 disjoint cycles or isomorphic to $K_3 + (n - 3)K_1$. 
Given an integer $k \geq 1$, how about $k$ disjoint cycles?

- K. Corradi and A. Hajnal, 1963
  For any positive integer $k$ and any graph $G$, if $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ has $k$ disjoint cycles.

Sharpness of degree condition is given by $G = K_{2k-1} + mK_1$. 

\[ G = K_{2k-1} + mK_1 \]
What is the degree condition for the disjoint cycles with the same length?

- G. A. Dirac, 1963
  For any positive integer $k$ and any graph $G$, if $n \geq 3k$ and $\delta(G) \geq (n + k)/2$, then $G$ contains $k$ disjoint triangles.

- The papers in this topic can be found in [Degree Conditions for the Existence of Vertex-Disjoint Cycles and Paths: A Survey, Graphs and Combinatorics (2018) 34:1 – 83. ]
For a digraph $D$, if $\delta^+(D) \geq 1$, then $D$ contains a cycle.

Given a positive integer $k$ and a digraph $D$, what is the degree condition for $k$ disjoint cycles in a digraph?

**Conjecture 2.1** [Bermond, Thomassen, J. Graph Theory 5 (1) (1981) 1-43.]

For $k \geq 1$ and any digraph $D$, if $\delta^+(D) \geq 2k - 1$, then $D$ contains $k$ disjoint cycles.

- Thomassen proved the case $k = 2$ in 1983 (Combinatorica);
- Lichiardopol et al. proved the case $k = 3$ in 2009 (SIAM Discrete Math.);
- **Conjecture 2.1 remains open for $k \geq 4$.**
Thomassen further proposed the conjecture on disjoint cycles with the same length.

**Conjecture 2.2** [Thomassen, Combinatorica 3 (3-4) (1983) 393-396.]

For each natural $k$ and any digraph $D$, there exists $g(k)$, if $\delta^+(D) \geq g(k)$, then $D$ contains $k$ pairwise disjoint cycles of the same length.

In 1996, Alon disproved the conjecture.


For every integer $r$, there exists a digraph $D$ such that $\delta^+(D) = r$, but $D$ contains no two edge-disjoint cycles of the same length (and hence, of course, no two vertex-disjoint cycles of the same length).
Although Conjecture 2.2 is not true for general digraphs, Lichiardopol believed that it is correct for tournaments. He raised the following conjecture.


For every tournament $T$, if $\delta^+(T) \geq (q - 1)k - 1$, then $T$ contains $k$ disjoint cycles of length $q$, where $q \geq 3$ and $k \geq 1$.

In the same paper, Lichiardopol himself proved that if both the minimum out-degree and in-degree are at least $(q - 1)k - 1$, i.e. $\delta^0(T) \geq (q - 1)k - 1$, then $T$ contains $k$ disjoint cycles of length $q$. 
In 2013, Jensen, Bessy and Thomassé proved Conjecture 2.4 for the case $q = 3$ (A special case of Bermond-Thomassen conjecture).

We confirm Lichiardopol’s Conjecture for $q \geq 4$.

**Main Theorem 1 [F. Ma, Y. Wang and Y, 2019+]**

Conjecture 2.4 is correct. That is, for any integer $q \geq 4$, $k \geq 1$ and every tournament $T$, if $\delta^+(T) \geq (q - 1)k - 1$, then it contains $k$ disjoint cycles of length $q$. 
We also improved Lichiardopol’s theorem on semi-degree condition significantly by proving the following theorem.

**Main Theorem 2** [F. Ma and Y, Applied Mathematics and Computation 347 (2019) 162-168.]

For integers \( k \geq 1 \) and \( q \geq 4 \), every tournament \( T \) with \( \delta^0(T) \geq (q - 1)k - 1 \) contains \( f(q)k - 2q \) disjoint cycles of length \( q \), where \( f(q) = \frac{6q^2 - 16q + 10}{3q^2 - 3q - 4} \).

**Note**

- \( f(q) > 1 \), when \( q \geq 4 \)
- \( f(q) \) tends to 2, when \( q \) tends to infinity.
3. Sketch of Main Theorem 1

Property 1. Every tournament has a hamiltonian path.

Property 2. Every strong tournament is vertex pancyclic.

- We prove Main Theorem by induction on $k$.
- When $k = 1$, then $\delta^+(T) \geq q - 2$.
  Let $P = (v_n \cdots v_1)$ be a hamiltonian path of $T$.

  Consider $v_1$, since $d^+(v_1) \geq q - 2$ and all its out-neighbours are on $P$, there exists $v_i$ with $i \geq q$ satisfying $v_1v_i \in A$.

- So there is a cycle $v_iv_{i-1}\cdots v_1v_i$ of length at least $q$. By property of pancyclicity, $T$ contains a $q$-cycle (a cycle of length $q$).
Suppose that Main Theorem is correct for \( k - 1 \), i.e. if \( \delta^+(T) \geq (q-1)(k-1)-1 \), then \( T \) contains \( k-1 \) disjoint \( q \)-cycles.

Now we consider the case \( k \).
Since in this case \( \delta^+(T) \geq (q-1)k-1 > (q-1)(k-1)-1 \), \( T \) contains \( k-1 \) disjoint \( q \)-cycles.

Using the following three auxiliary theorems, we may show that \( T \) contains \( k \) disjoint \( q \)-cycles,

**Theorem 3.1**
Let \( q \geq 9 \) and \( k \leq q + 1 \). For every collection \( \mathcal{F} \) of \( k-1 \) disjoint \( q \)-cycles of \( T \), there exists a collection of \( k \) disjoint \( q \)-cycles in \( T \) which intersects \( T \setminus \mathcal{F} \) on at most \( 3q \) vertices.
Theorem 3.2
Let $q \geq 10$ and $k \geq 3.3098\sqrt{q}$. For every collection $\mathcal{F}$ of $k - 1$ disjoint $q$-cycles of $T$, there exists a collection of $k$ disjoint $q$-cycles in $T$ which intersects $T \setminus \mathcal{F}$ on at most $3q$ vertices.

- Since $3.3098\sqrt{q} \leq q + 2$ when $q \geq 8$, Theorems 3.1 and 3.2 imply Main Theorem for $q \geq 10$.

Theorem 3.3
When $4 \leq q \leq 9$, we can refine Theorem 3.2 with "$|\mathcal{F} \cap (T \setminus \mathcal{F})| \leq 3q - 6$".
That is, For every collection $\mathcal{F}$ of $k - 1$ disjoint $q$-cycles of $T$, there exists a collection of $k$ disjoint $q$-cycles in $T$ which intersects $T \setminus \mathcal{F}$ on at most $3q - 6$ vertices.
3. Sketch of Theorem 3.1

Let $\mathcal{F} = \{C_1, \ldots, C_{k-1}\}$ be a collection of $k - 1$ disjoint $q$-cycles, $P = (u_r \cdots u_2 u_1)$ be a hamiltonian path of $T \setminus \mathcal{F}$.

We design the following pseudo algorithm.

**step 0.** Set $\mathcal{F} := \{C_1, \ldots, C_{k-1}\}$, $P := (u_r \cdots u_2 u_1)$ and $h := 0$.

Let $C = (u_{j-1}u_{j-2} \cdots u_1u_{j-1})$ be the longest cycle of $T[V(P)]$ through vertex $u_1$, if it exists.

**step 1.** If $C$ does not exist, then we construct a collection $\mathcal{F}'$ of $k - 1$ disjoint $q$-cycles and a new hamiltonian path $P' = (v_r \cdots v_2 v_1)$ in $T \setminus \mathcal{F}'$ such that $|V(\mathcal{F}') \cap V(P)| \leq 2$ and $T \setminus \mathcal{F}'$ contains a cycle $C'$ through vertex $v_1$. Set $\mathcal{F} := \mathcal{F}'$, $P := P'$, $C := C'$ and $h := h + 1$. Otherwise, go to step 2.
3. Sketch of Theorem 3.1

**step 2.** Construct a collection \( \mathcal{F}' \) of \( k-1 \) disjoint \( q \)-cycles and a new hamiltonian path \( P' = (v_r \cdots v_2v_1) \) in \( T \setminus \mathcal{F}' \) such that \(|V(\mathcal{F}') \cap V((P))| \leq 2\) and \( T \setminus \mathcal{F}' \) contains a cycle \( C' \) with \(|C'| \geq |C|+1\) through vertex \( v_1 \). Set \( \mathcal{F} := \mathcal{F}' \), \( P := P' \), \( C := C' \) and \( h := h + 1 \).

![Diagram](image)

**step 3.** If \(|C| \leq q - 1\), then do step 2 recursively until \(|C| \geq q\). Otherwise, output \( \mathcal{F} \cup \{C\} \) and \( h \).
3. Sketch of Theorem 3.1

- After step 3, $|C| \geq q$, by pancyclicity, there is a $q$-cycle $C'$. This cycle and $\mathcal{F}$ form a collection of $k$ disjoint $q$-cycles.

- Now we prove that the vertex number of $|(V(\mathcal{F} \cup \{C'\})) \cap V(P)|$ is at most $3q$ by using the algorithm.

- At each iteration of step 1 and step 2, we add at most two vertices outside $\{C_1, \ldots, C_{k-1}\}$ into $\mathcal{F}$. It follows by $h \leq q - 2$ that

$$|V(\mathcal{F}) \cap V((T \setminus \{C_1, \ldots, C_{k-1}\}))| \leq 2h \leq 2q - 4.$$ 

Therefore,

$$|(\mathcal{F} \cup \{C'\}) \cap (T \setminus \{C_1, \ldots, C_{k-1}\})| \leq 2q - 4 + q = 3q - 4.$$ 

Thus Theorem 3.1 holds.
Sketch of Theorem 3.2

Let $\mathcal{F} = \{C_1, \ldots, C_{k-1}\}$ be a collection of $k-1$ disjoint $q$-cycles and let $P = (u_r \cdots u_2 u_1)$ be a hamiltonian path of $T \setminus \mathcal{F}$. Partition $P$ by letting $U_1 = \{u_1, \ldots, u_{q+1}\}$, $S = \{u_{q+2}, \ldots, u_{4q-5}\}$ and $U_2 = V(P) \setminus (U_1 \cup S)$. That is, $U_1$ is the set of the last $q+1$ vertices on $P$ and $S$ is the last $3q-6$ vertices on $P \setminus U_1$. 

![Diagram of partition of P with vertices U1, U2, and S](image-url)
3. Sketch of Theorem 3.2

Denote by $I$ the set of $q$-cycles that receive at least $q^2$ arcs each from $U_1$, by $O$ the set of $q$-cycles that send at least $6q-1$ arcs each to $U_2$ and $R = \mathcal{F} \setminus (I \cup O)$. Furthermore, $i, o$ and $r$, respectively, denote the size of $I, O$ and $R$. 

Fig. 3 Partition of $F$
3. Sketch of Theorem 3.2

Now we estimate the lower and upper bound of the number of arcs leaving from $\mathcal{F} \setminus \mathcal{O} = \mathcal{I} \cup \mathcal{R}$.

Fig. 3 Partition of $F$
3. Sketch of Theorem 3.2

**First**, since \( \delta^+(T) \geq (q - 1)k - 1 \),

\[
d^+(\mathcal{I} \cup \mathcal{R}) \geq q(i + r)((q - 1)k - 1) - \frac{1}{2}q(i + r)(q(i + r) - 1).
\]

**On the other hand**, we bound the number of arcs from \( \mathcal{I} \) to \( \mathcal{O} \) and \( \mathcal{R} \) to \( \mathcal{O} \), from \( \mathcal{I} \) to \( U_2 \) and \( \mathcal{R} \) to \( U_2 \), from \( \mathcal{I} \cup \mathcal{R} \) to \( S \) and \( U_1 \).

\[
d^+(\mathcal{I} \cup \mathcal{R}) \leq (q^2 - q + 2)i_o + q^2r_o + (3q - 1)i + (6q - 2)r + (3q - 6)q(i + r)
\]
\[
+q(i + r)(q + 1) - \alpha + (q^2 - q - 2)o,
\]

where \( \alpha = (q + 1)((q - 1)k - 1) - \frac{1}{2}(q + 1)q - \frac{1}{2}(q - 2)(q - 3) \).
3. Sketch of Theorem 3.2

So we get
\[ ao^2 + bo + c < 0, \] (1)

where
\[ a = \frac{1}{2}q^2 + q - 2, \quad b = ((q - q^2)k) + (3q^2 + \frac{13}{2}q - 9) \]

and
\[ c = (\frac{1}{2}q^2 - q)k^2 + (1 - \frac{1}{2}q - 3q^2)k + (\frac{5}{2}q^2 + \frac{11}{2}q - 25). \]

Obviously, \( a = \frac{1}{2}q^2 + q - 2 > 0. \) Inequality (1) admits solution for \( o \) only if
\[ \Delta = (-2q^3 + 9q^2 - 8q)k^2 + (6q^3 + 7q^2 - 26q + 8)k + 4q^4 + 18q^3 + \frac{145}{4}q^2 + 27q - 119 > 0. \] (2)
3. Sketch of Theorem 3.2

Note that (2) is a quadratic inequality for $k$. Since $-2q^3 + 9q^2 - 8q < 0$, the inequality (2) has a solution only if

$$k < g(q),$$

where

$$g(q) = \frac{6q^3 + 7q^2 - 26q + 8 + f(q)}{4q^3 - 18q^2 + 16q},$$

and

$$f(q) = \sqrt{32q^7 + 36q^6 - 146q^5 - 776q^4 - 1032q^3 + 5936q^2 - 4224q + 64}.$$ 

It follows by $q \geq 10$ that $k < 3.3098\sqrt{q}$, i.e. the inequality (2) has a solution only if $k < 3.3098\sqrt{q}$. This contradicts $k \geq 3.3098\sqrt{q}$. So Theorem 3.2 is proved. □
4. Refine Theorem 3.2 for $4 \leq q \leq 9$

Let $U_1 = \{u_1, \ldots, u_q\}$, $S = \{u_{q+1}, \ldots, u_{4q-10}\}$ and $U_2 = V(P) \setminus (U_1 \cup S)$ (If $|P| < 4q - 10$, then let $U_1 = \{u_1, \ldots, u_q\}$, $S = V(P) \setminus U_1$).

Define

$$I = \{C \in \mathcal{F} \mid d^+(U_1, C) \geq q(q - 1) + 1\},$$

$$O = \{C \in \mathcal{F} \mid d^+(C, U_2) \geq 6q - 13\} \quad \text{and} \quad \mathcal{R} = \mathcal{F} \setminus (I \cup O).$$

- Let $C$ be a $q$-cycle. If there is a $q$-matching $M$ from $U_1$ to $C$, then $d^+(C, S) \leq \frac{9}{4}q^2 - \frac{29}{4}q$.

Similar, estimate the lower and upper bound of $d^+(\mathcal{F} \setminus O)$. We get all the possible cases: if $q = 4$, then $1 \leq k \leq 4$; if $9 \geq q \geq 5$, then $1 \leq k \leq 5$.

From the following statement, we finish the proof of the case $4 \leq q \leq 9$.

- Let $k$ be an integer with $k \leq 5$. If there exist two cycles $C_1, C_2 \in \mathcal{F}$ such that $d^+(\{u_1, u_2\}, C_i) \geq 2q - 1$ for $i = 1, 2$, then we can extend $\mathcal{F}$. 
4. Related Results and Open Problems

– Cycle Factor in Digraphs

• For a subset $W \subseteq V(D)$, define

$$
\delta^+(W) = \min \{ d^+_D(v) : v \in W \},
$$

$$
\delta^-(W) = \min \{ d^-_D(v) : v \in W \}.
$$

• **The minimum semi-degree of $W$ in $D$:**

$$
\delta^0(W) = \min \{ \delta^+(W), \delta^-(W) \}.
$$

**Theorem 4.1 [Y. Wang and Y, 2019+]**

Suppose that $D$ is a digraph with order $n$ and $W \subseteq V(D)$. If

$$
\delta^0(W) \geq (3n - 3)/4,
$$

then for any $k$ positive integers $n_1, \ldots, n_k$ with $n_i \geq 2$ for all $i$ and

$$
\sum_{i=1}^{k} n_i \leq |W|,
$$

$D$ contains $k$ disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap W| = n_i$ for each $i$. 
**A directed version of the Aigner-Brandt Theorem, when** \( W = V(D) \) [J. Lond. Math. Soc. 1 (1993) 39-51]: If \( \delta(G) \geq (2n - 1)/3 \), then \( G \) contains \( k \) disjoint cycles of length \( n_1, \ldots, n_k \), respectively. \( (n \geq \sum_{i=1}^{k} n_i \) and \( n_i \geq 3 \) for all \( i \))

**Sharp** (in some sense)

\[
D_1 : U = X = K_{4k-1}^*, Y = Z = K_{4k}^*
\]

\[
D_2 : X = K_{2k-1}^*, Y \text{ is an independent vertex set of order } k + 1.
\]
Conjecture 4.2 [Y. Wang and Y, 2019+]

The minimum semi-degree in Theorem 4.1 can be improved to $2n/3$ when $n_i \geq 3$ for all $i$.

The degree condition is best possible by $D_2$.

It is supported by the following conjecture.

Conjecture 4.3 [Czygrinow, Kierstead and Molla, Eur. J. Combin. 42 (2013) 1-14]

If $n = 3k$ and $\delta^0(D) \geq 2k$, then $D$ contains $k$ disjoint $\triangle$s.
Remark for Theorem 4.1.

- When $k = 1$, $\delta^0(W) \geq (3n - 3)/4 \implies \delta^0(W) \geq \frac{n}{2}$

- Let $\lambda = \sum_{i=1}^{k} n_i$.

  If $n \geq 2\lambda$, then $\delta^0(W) \geq (3n - 3)/4 \implies \delta^0(W) \geq \frac{n}{2} + \lambda - 1$. 
Thanks for your attention!