Critical permutation sets for generalized signed graph colouring

齊豪
Academia Sinica

Assume G is a graph and k is a positive integer. We view G as a symmetric digraph, in which each edge uv of G is replaced by a pair of opposite arcs $e = (u, v)$ and $e^{-1} = (v, u)$. Let S be a set of permutations on $[k]$ that is inverse closed. An S-signature of G is a mapping $\sigma : E(G) \to S$ for which $\sigma_{e^{-1}} = \sigma_e^{-1}$. The pair (G, σ) is called a generalized signed graph. A k-colouring of (G, σ) is a mapping $\varphi : V(G) \to [k]$ such that for each arc $e = (u, v)$, $\sigma_e(\varphi(u)) \neq \varphi(v)$. We say G is S-k-colourable if for any S-signature σ of G, (G, σ) is k-colourable. If $S = \{id\}$, then S-k-colourable is the same as k-colourable. If $S = S_k$, then S-k-colourable is equivalent to DP-k-colourable. For other inverse closed sets S of permutations, S-k-colourability reveals a complex hierarchy of colourability of graphs. We say an inverse closed subset S of S_k is critical if for any inverse closed subset S' containing S as a proper subset, there is a graph G which is S-k-colorable but not S'-k-colorable. For a set X, denote by S_X the symmtric group of all permutations on X. In this paper, we prove the following results: Assume $[k]$ is the disjoint union of $X_1, X_2, \ldots, X_q$. If $S = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_q$, where for each i, either $\Gamma_i = S_{X_i}$ or $|X_i| = 3$ and Γ_i is the subgroup of S_{X_i} generated by a cyclic permutation of X_i, then S is critical.