The signature of two generalization of line graphs

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Preliminaries

- Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. Its adjacency matrix $A(G) = (a_{ij})$ is defined as $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$, if v_i is adjacent to v_j; and $a_{ij} = 0$, otherwise.

- The eigenvalues of $A(G)$ are said to be the *eigenvalues of the graph G*. The numbers of positive and negative eigenvalues of the graph G are called *positive inertia index* and *negative inertia index* of the graph G, and are denoted by $p(G)$ and $n(G)$, respectively. The *signature* of G, denoted by $s(G)$, is defined to be the difference $p(G) - n(G)$.

Preliminaries

- A connected graph has exactly one positive eigenvalue if and only if it is complete multipartite.
- (see [1]) Let G be a graph. Then we have $\mid p(G) - n(G) \mid \leq c_1(G)$, where $c_1(G)$ is the number of odd cycles of G.
- (see [2]) Denote by $m(G)$ the *matching number* of graph G, and $c(G) = \mid E(G) \mid - \mid V(G) \mid + \theta(G)$ the *cyclomatic number* of G, then $m(G) - c(G) \leq p(G) \leq m(G) + c(G)$.

H. Ma, W. Yang, S. Li, Positive and negative inertia index of a graph, Linear Algebra Appl. 438 (2013) 331–341.

Y.Z. Fan, L. Wang, Bounds for the positive and negative inertia index of a graph, Linear Algebra Appl., 522(2017)15-27.

Preliminaries

Let $c_3(G)$ and $c_5(G)$ denote respectively the number of cycles having length $4k+3$ (or length 3 modulo 4) and the number of cycles having length $4k+5$ for some integers $k \geq 0$ (or length 1 modulo 4).

Conjecture 1

The inequality $-c_3(G) \leq s(G) \leq c_5(G)$ possibly holds for any simple graph G.

H. Ma, W. Yang, S. Li, Positive and negative inertia index of a graph, Linear Algebra Appl. 438 (2013) 331–341.

Preliminaries

The conjecture was proved holds for trees, unicyclic, bicyclic graphs, line graphs and all power graphs G^k where $k \geq 2$.

- H. Ma, W. Yang, S. Li, Positive and negative inertia index of a graph, Linear Algebra Appl., 438 (2013) 331-341.
- X.B. Ma, D.I. Wong, M. Zhu, The positive and the negative inertia index of line graphs of trees, Linear Algebra Appl., 439(10)(2013)3120-3128.
- L. Wang, Y.Z. Fan, The signature of line graphs and power trees, Linear Algebra and its Applications, 448 (2014) 264-273.
- X. B. Ma, X. Y. Geng, Signature of power graphs, Linear Algebra and its Applications, 545(2018)139-147.

Claw-free graphs

The *line graph* of a graph G, denoted by L_G, is the graph whose vertex set is $E(G)$, where two vertices of L_G are adjacent if and only if the corresponding edges are incident in G.

A graph G is *claw-free* if no vertex has three pairwise nonadjacent neighbours, i.e., $K_{1,3}$ is not an induced subgraph of G.

Lemma 1

A graph is a line graph if and only if it has no induced subgraph isomorphic to one of the graphs $H_1,\ H_2,\ldots,H_9$ from Figure 1.

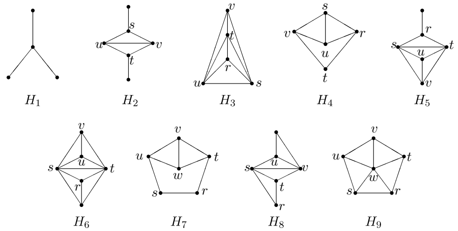

Figure: Forbidden induced subgraphs for line graphs

D.Cvetkovic, P. Rowlinson, S. Simic,Spectral Generalizations of Line Graphs On Graphs with Least Eigenvalue -2, Book DOI: http://dx.doi.org/10.1017/CBO9780511751752.

Claw-free graphs

Lemma 2

Let G be a graph with $S \subset V(G)$, and let $H = G - S$. Then $\mid s(G) - s(H) \mid \leq k$, where k is the number of vertices in S.

Lemma 3

Let G be a graph with H as a subgraph. Then G admits Conjecture 1 if the following conditions are all satisfied:

1. *H admits Conjecture 1;*
2. *$\mid s(G) - s(H) \mid \leq k$ with k an nonnegative integer;*
3. *$c_i(G) \geq c_i(H) + k$ for $i \in \{3, 5\}$.*

X. B. Ma, X. Y. Geng, Signature of power graphs, Linear Algebra and its Applications, 545(2018)139-147.

Claw-free graphs

Theorem 4

If G is a claw-free graph, then $-c_3(G) \leq s(G) \leq c_5(G)$.

Proof Assume by contradiction that in the set of all counterexamples, G has the minimum number of vertices. We claim that G has an induced subgraph isomorphic to one of the graphs $H_2, \ldots, H_9$ from Figure 1. Otherwise, G is a line graph (by Lemma 1), and thus $-c_3(G) \leq s(G) \leq c_5(G)$ follows from Lemma **??**. This is a contradiction.
If there exists a vertex x in G which is contained in two cycles with length 1 and 3 modulo 4, respectively, then deleting the vertex x will decrease $c_3(G)$ and $c_5(G)$, i.e., $c_3(G) \geq c_3(G - x) + 1$ and $c_5(G) \geq c_5(G - x) + 1$. Clearly $G - x$ is also claw-free, and since G is a minimum counterexample, it follows that $-c_3(G - x) \leq s(G - x) \leq c_5(G - x)$. We easily have $-c_3(G) \leq s(G) \leq c_5(G)$, which contradicts the fact that G is a counterexample.
If G has an induced subgraph isomorphics to H_j, $j \in \{3, 4, \ldots, 9\}$, we can take $x = v$, where v is the vertex as shown in Figure 1 (*vusrt* is a pentagon) and get a contradiction.

Claw-free graphs

Next, we consider the case that H_2 is an induced subgraph of G.
Let v be one of the degree 3 vertices in H_2 (as shown in Figure 1). If $d_G(v) \geq 4$, there exists a neighbour of v in G which is different from u, s and t, say w. Note that since G is claw-free, either s or t is adjacent to w; otherwise, the subgraph induced by v, w, s and t is a claw. Without loss of generality, we may assume that $sw \in E(G)$. It is easy to check that svw and $vtusw$ are a triangle and pentagon, so by taking $x = v$, we can get a contradiction. Similarly, if $d_G(u) \geq 4$, we can take $x = u$ and obtain a contradiction.
In the remainder of the proof we assume that $d_G(u) = d_G(v) = 3$.
The adjacency matrix of G can be written as

$$A(G) = \begin{pmatrix} \mathbf{J}_2 - \mathbf{I}_2 & \mathbf{J}_2 & \mathbf{O}_{2\times(n-4)} \\ \mathbf{J}_2 & \mathbf{O}_{2\times 2} & \mathbf{X} \\ \mathbf{O}_{(n-4)\times 2} & \mathbf{X}^T & \mathbf{C} \end{pmatrix},$$

which is congruent to the matrix

$$M = \begin{pmatrix} \mathbf{N} & \mathbf{O}_{2\times 2} & \mathbf{O}_{2\times(n-4)} \\ \mathbf{O}_{2\times 2} & -2\mathbf{J}_2 & \mathbf{X} \\ \mathbf{O}_{(n-4)\times 2} & \mathbf{X}^T & \mathbf{C} \end{pmatrix},$$

where $N = (\; 0 \quad 1 \;)$

Claw-free graphs

Let $Q = \begin{pmatrix} -2\mathbf{J}_2 & \mathbf{X} \\ \mathbf{X}^T & \mathbf{C} \end{pmatrix}$, then $s(Q) = s(M) = s(G)$. Observe that the adjcency matrix of $G - u - v$ is $\begin{pmatrix} \mathbf{O}_{2\times 2} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{C} \end{pmatrix}$, so we have $Q = A(G - u - v) + \begin{pmatrix} -2\mathbf{J}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$. By direct calculation, the eigenvalues of $\begin{pmatrix} -2\mathbf{J}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$ are $0^{(n-3)}$, -4, where n is the order of G. By Lemma **??**, $s(G) = s(Q) \leq s(G - u - v)$. In view of the minimality of G, the subgraph $G - u - v$ admits the conjecture, i.e.,

$$-c_3(G - u - v) \leq s(G - u - v) \leq c_5(G - u - v).$$

So we can conclude that $s(G) \leq s(G - u - v) \leq c_5(G - u - v) \leq c_5(G)$. On the other hand, Lemma 2 implies that $s(G) \geq s(G - u - v) - 2 \geq -c_3(G)$. Consequently, we have $-c_3(G) \leq s(G) \leq c_5(G)$, which contradicts to the fact that G is a counterexample. In all cases we have a contradiction, and the desired result is proved.

Graphs with least eigenvalue $\geq$-2

Lemma 5 (see [1])

Let G be a connected graph with least eigenvalue ≥ -2. Then one of the following holds:

1. *G is a generalized line graph;*
2. *G has a $(2, 1, 0)$-representation by roots of E_8. The number n of vertices, and the average degree $\overline{d}$ are restricted by $n \leq \min\{36, 2\overline{d} + 8\}$. Moreover, every vertex has degree at most 28.*

P. J. Cameron, J. M. Goethals, J. J. Seidel, and E. E. Shult, Line graphs, root systems and elliptic geometry, J. Algebra, 43 (1976), 305-327.

Graphs with least eigenvalue $\geq$-2

The *cocktail party graph* $CP(n)$ is the unique regular graph with $2n$ vertices of degree $2n-2$; it is obtained from K_{2n} by deleting a perfect matching.

Definition 6

Let H be a graph with vertex set $\{v_1, \ldots, v_n\}$, and let $a_1, \ldots, a_n$ be non-negative integers. The generalized line graph $G = L(H; a_1, \ldots, a_n)$ consists of disjoint copies of L_H and $CP(a_1), CP(a_2), \ldots, CP(a_n)$ along with all edges joining a vertex $\{v_i,\ v_j\}$ of L_H with each vertex in $CP(a_i)$ and $CP(a_j)$.

Lemma 7

If G is a generalized line graph, then $-c_3(G) \leq s(G) \leq c_5(G)$.

Graphs with least eigenvalue $\geq$-2

A graph of type (2) is called *exceptional* (it is connected, has the least eigenvalue greater than or equal to -2, and is not a generalized line graph). According to Lemma 7 and Lemma 5, if we can prove exceptional graphs admit the conjecture, then we can get our main result in this section which is as follows.

Theorem 8

If G is a graph with least eigenvalue ≥ -2, then $-c_3(G) \leq s(G) \leq c_5(G)$.

It is almost an impossible task to generate all exceptional graphs, and to verify whether they admit the conjecture or not. Accordingly we can try to list some necessary conditions to restrict the minimum counterexamples (counterexamples which have the minimum number of vertices) among all exceptional graphs. If there is no such minimum counterexamples, then the result follows.

Lemma 9

The maximum degree of G, say $\Delta(G)$, is less than 8.

Graphs with least eigenvalue $\geq$-2

With the help of the software Sagemath, we find out all connected graphs which satisfy the following four conditions, and we call them *candidate graphs*:

1. have an induced subgraph isomorphic to one of the graphs $G^{(12)}, G^{(13)}, \ldots, G^{(17)}$;
2. the least eigenvalues are greater than or equal to -2;
3. any triangle and any pentagon are vertex disjoint;
4. the maximum degrees are less than or equal to 7.

we present an algorithm to generate all candidate graphs (See function genCanGs() in the Appendix) and establish the following relevant facts. There are exactly 197 candidate graphs and their orders are at most 11.

Chordal graphs and cographs

1. A graph G is said to be *chordal* if every cycle of length $\ell \geq 4$ in G has a chord, i.e., there is an edge $e \in E$ connecting two nonconsecutive vertices of the cycle.
2. P_4-free graphs are called *cographs*.

Theorem 10

If G is a chordal graph or a cograph, then $-c_3(G) \leq s(G) \leq c_5(G)$.

Thank you!