On judicious bipartitions of directed graphs

Jianfeng Hou

Fuzhou University

Email: jfhou@fzu.edu.cn

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Outline

Max-Cut problem

Let G be a graph. A bipartition of G, denoted by (V_1, V_2), is a bipartition of $V(G)$ with $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.

If it satisfies $||V_1| - |V_2|| \leq 1$, then we call it a bisection.

The size of (V_1, V_2), denoted by $e(V_1, V_2)$, is the number of edges with one end in V_1 and the other in V_2.

The famous Max-Cut problem is to find a bipartition (V_1, V_2) of a given graph G that maximizes $e(V_1, V_2)$.

Brief analysis

Suppose that G has m edges and chromatic number k.

Let $(V_1, \ldots, V_k)$ be a k-coloring of G. Let G' be the complete multigraph obtained from G by identifying each V_i into a single vertex.

Then every bipartition of G' yields a bipartition of G with the same size.

Continued

By randomly partition $V(G)$ into two subsets of size $\lfloor k/2 \rfloor$ and $\lceil k/2 \rceil$, the expected number of edges between the two subsets is

$$\frac{\lfloor k/2 \rfloor \times \lceil k/2 \rceil}{\binom{k}{2}} m \geq \frac{(k^2-1)/4}{(k^2-k)/2} m = \frac{m}{2} + \frac{m}{2k}.$$

Since $m \geq \binom{k}{2}$, we have

$$k \leq \sqrt{2m + \frac{1}{4}} + \frac{1}{2}.$$

Edwards's bound

This implies that G has a bipartition (V_1, V_2) such that

$$e(V_1, V_2) \geq \frac{m}{2} + \frac{1}{4}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right),$$

and this bound is best possible for K_{2n+1}.

C.S. Edwards, Some extremal properties of bipartite graphs, *Canad. J. Math.* **3** (1973) 475–485.

C.S. Edwards, An improved lower bound for the number of edges in a largest bipartite subgraph, *Proc. 2nd Czechoslovak Symposium on Graph Theory* (1975) 167–181.

Triangle-free graphs

An natural direction of the Max-Cut problem is to bound the Max-Cut of graphs without a specific subgraph H. The Max-Cut problem of triangle free graphs was posed by Erdős.

Theorem

Every triangle free graph G with m edges admits a bipartition (V_1, V_2) such that

$$e(V_1, V_2) \geq m/2 + cm^{2/3},$$

for some $c > 0$.

P. Erdős, Problems and results in graph theory and comobinatorial analysis, In Graph Theory and Related Topics (Proc. Conf. Waterloo, 1977), Academic Press, New York (1979) 153–163.

Shearer's bound

Let $d_1, \ldots, d_n$ be the degree sequence of a triangle free graph G with n vertices and m edges. Shearer showed that G has a bipartition of size at least

$$m/2 + \frac{1}{8\sqrt{2}} \sum_{i=1}^{n} \sqrt{d_i}.$$

It follows as a corollary that G has a bipartition of size at least

$$m/2 + cm^{3/4},$$

for some $c > 0$.

J. Shearer, A note on bipartite subgraphs of triangle-free graphs, Random Struct. Alg. 3 (1992) 223–226.

Alon's bound

In 1996, Alon proved

Theorem

Every triangle free graph G with m edges admits a bipartition (V_1, V_2) *such that*

$$e(V_1, V_2) = m/2 + \Theta(m^{4/5}).$$

N. Alon, Bipartite subgraphs, Combinatorica 16 (1996) 301–311.

Large girth

For $r \geq 3$, let C_r denote a cycle of length r. Alon, Bollobás, Krivelevich and Sudakov considered the maximum cut of graphs G with large girth, and proved that every graph G with m edges and girth at least r admits a bipartition (V_1, V_2) such that

$$e(V_1, V_2) \geq m/2 + c(r)m^{\frac{r}{r+1}}$$

for some $c(r) > 0$.

N. Alon, B. Bollobás, M. Krivelevich, B. Sudakov, Maximum cuts and judicious partitions in graphs without short cycles, J. Combin. Theory Ser. B 88 (2003) 329–346.

Even cycle

Alon, Krivelevich and Sudakov considered graphs without even cycles and proved that

Theorem

For every odd $r \geq 5$, and a C_{r-1}-free graph G with m edges, there is a positive constant $c(r)$ such that G has a bipartition of size at least

$$m/2 + c(r)m^{\frac{r}{r+1}}.$$

N. Alon, M. Krivelevich and B. Sudakov, Maxcut in *H*-free graphs, Combin. Probab. Comput. 14 (2005), 629–647.

Conjecture

In the same paper, they conjectured that the lower bound holds for graphs without odd cycles.

Conjecture

For every even integer $r \geq 4$, and a C_{r-1}-free graph G with m edges, there is a positive constant $c(r)$ such that G has a bipartition of size at least

$$m/2 + c(r)m^{\frac{r}{r+1}}.$$

So far, this conjecture was confirmed for $r = 4$.

Our result

We considered the conjecture and established the following theorem.

Theorem (H., Zeng, 2018, ARS Math. Contemp.)

For every even integer $r > 4$, and a C_{r-1}-free graph G with m edges, there is a positive constant $c(r)$ such that G has a bipartition of size at least

$$m/2 + c(r)\left(m^r \log^4 m\right)^{\frac{1}{r+2}}.$$

5-Cycles free

For C_5-free graphs, we have

Theorem (H., Zeng, 2018, ARS Math. Contemp.)

For each $s \geq 2$, let G be a $\{K_{2,s}, C_5\}$-free graph with m edges. Then G has a bipartition of size at least

$$m/2 + c(s)m^{6/7}$$

for a positive constant $c(s)$.

General graph

It is noted that for every fixed graph H there exist positive constants $\epsilon = \epsilon(H)$ and $c = c(H)$ satisfying that:

if a H-free graph G has m edges, then G has a bipartition of size at least

$$m/2 + cm^{1/2+\epsilon}.$$

A conjecture

Alon, Bollobás, Krivelevich and Sudakov posed the following conjecture.

Conjecture

Let H be a fixed graph, and let G be a graph with m edges. If G is H-free, then G has a bipartition of size at least

$$\frac{m}{2} + \Omega(m^{3/4+\epsilon}).$$

for some $\epsilon > 0$.

N. Alon, B. Bollobás, M. Krivelevich, B. Sudakov, Maximum cuts and judicious partitions in graphs without short cycles, J. Combin. Theory Ser. B 88 (2003) 329–346.

Clearly, it suffices to prove this conjecture for complete graphs H. We considered K_{k+1}-free graph and proved that

Theorem (H., Zeng, 2017, Bull. Aust. Math. Soc.)

For any fixed integer $k \geq 2$, every K_{k+1}-free graph with m edges admits a bipartition of size at least

$$\frac{m}{2} + c(k) m^{\frac{k}{2k-1}} \left(\frac{\log^2 m}{\log \log m} \right)^{\frac{k-1}{2k-1}}.$$

for some $c(k) > 0$.

Max-Bisection

The Max-Bisection problem is that given a graph G, find a bisection (V_1, V_2) of G that maximizes $e(V_1, V_2)$.

Compared to bipartitions, bisections are much more complicated to analyze.

For example, Edwards' bound implicitly implies that a connected graph G with n vertices and m edges admits a bipartition (V_1, V_2) with

$$e(V_1, V_2) \geq m/2 + (n-1)/4.$$

Unfortunately, this result cannot transfers directly to bisections.

Example

Let $K_{k,n-k}$ be the complete bipartite graph with $k \le n/2$.

It is easy to see that, the size of the maximum bisection of $K_{k,n-k}$ is $m/2 + k^2/2$ if n is even, and it is $m/2 + (k^2 + k/)2$ if n is odd, where $m = k(n-k)$.

So, for bisections, we cannot get a bound greater than $(m+1)/2$ for every graph G with m edges.

Maximum matching

Let G be a graph with m edges, and let M be a maximum matching of G.

Then an easy analysis, given by Xu, Yan and Yu, shows that G has a bisection of size at least

$$\frac{m + |M|}{2}.$$

B. Xu, J. Yan, X. Yu, A note on balanced bipartitions, Discrete Math. 310 (2010) 2613–2617.

Tight graph

A connected graph T is tight if

- for every vertex $v \in V(T)$, $T - v$ contains a perfect matching, and
- for every vertex $v \in V(T)$ and every perfect matching M of $T - v$, no edge in M has exactly one end adjacent to v.

Lu, Wang and Yu showed that a connected graph G is tight iff every block of G is an odd clique.

C. Lu, K. Wang, X. Yu, On tight components and anti-tight components, Graphs and Combinatorics 31 (2015) 2293–2297.

Tight bound

Lee, Loh and Sudakov gave a lower bound of Max-Bisection of graphs with respect to the tight components and maximum degree by showing

Theorem

Let G be a graph with m edges and maximum degree Δ. If G has τ tight components, then there is a bisection of size at least

$$\frac{m}{2} + \frac{n - \max\{\tau, \Delta - 1\}}{4}.$$

The vertex-disjoint copies of a triangle and the star $K_{1,n-1}$ show that the theorem is tight in both parameters τ and Δ.

C. Lee, P. Loh, B. Sudakov, Bisections of graphs, J. Combin. Theory Ser. B 103 (2013) 590–629.

Our result

The following result showed a similar bound on bisections of graphs without short cycles.

Theorem (Fan, H., Yu, 2018, CPC)

Let G be a graph with n vertices, m edges and minimum degree $\delta \geq 2$, and without 4-cycles. If $\delta(G)$ is even, then G has a bisection of size at least

$$\frac{m}{2} + \frac{n-1}{4} - \frac{|M|}{2\delta},$$

where M is a maximum matching in G. Moreover, if the girth $g(G) \geq 5$, then G has a bisection of size at least

$$\frac{m}{2} + \frac{n-1}{4}.$$

Generalized results (1)

Note that the distance between two triangles is the length of a shortest path between their vertices.

Jin and Xu generalized the above result by giving the following theorem.

Theorem

Let $l \geq 2$ be an integer, and let G be a connected graph with minimum degree at least 2 and without $K_{2,l}$. Then, G admits a bisection (V_1, V_2) with $e(V_1, V_2) \geq m/2 + (n - l + 1)/4$ if either G is Eulerian, or G has no triangles at distance 2.

J. Jin and B. Xu, Bisection of graphs without $K_{2,l}$, Discrete Appl. Math. 259 (2019) 112–118.

Generalized results (2)

A (l, r)-fan is a graph on $(r-1)l+1$ vertices consisting of l cliques of order r that intersect in exactly one common vertex.

Theorem (H., Yan, 2019, DM)

Let $l \geq 2$ be an integer, and let G be a connected graph with m edges and minimum degree at least 2. If G contains neither $K_{2,l}$ nor $(l, 4)$-fan, then G admits a bisection (V_1, V_2) with

$$e(V_1, V_2) \geq \frac{m}{2} + \frac{n-l+1}{4}.$$

Corollary

When $l = 2$, we have the following corollary.

Corollary

Every connected graph G with m edges and minimum degree at leat 2, and without 4-cycles admits a bisection of size at least

$$\frac{m}{2} + \frac{n-1}{4}.$$

Bondy-Simonovits Theorem

The following result of Bondy and Simonovits gives the maximum number of edges in graphs without cycles of a given even length.

Theorem

Let $l \geq 2$ be an integer and let G be a graph with n vertices. If G contains no cycle of length $2l$, then the number of edges in G is at most $100ln^{1+1/l}$.

A. Bondy, M. Simonovits, Cycles of even length in graphs, J. Combin. Theory Ser. B 16 (1974) 97–105.

New bound

Combining Corollary 8 and Theorem 9, we have

Theorem

Let G be a graph with m edges and minimum degree $\delta \geq 2$. If G contains no 4-cycle, then G has a bisection of size at least

$$m/2 + cm^{2/3},$$

for some constant $c > 0$.

Definition

Given a digraph D, a digraph version of the Max-Cut problem is to find a partition (V_1, V_2) of D that maximizes

$$\min\{e(V_1, V_2), e(V_2, V_1)\}.$$

Here, $e(V_1, V_2)$ is the number of arcs xy with $x \in V_1$ and $y \in V_2$.

Such a problem are called Judicious Partition Problem by Bollobás and Scott.

B. Bollobás, A.D. Scott, Problems and results on judicious partitions, Random Struct. Alg. 21 (2002) 414–430.

Scott's problem

Scott posed the following natural problem:

What is the maximum constant c_d such that every digraph D with m arcs and minimum outdegree d admits a bipartition (V_1, V_2) satisfying that

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geq c_d m?$$

A.D. Scott, Judicious partitions and related problems, Surveys in Combinatorics 327 (2005) 95–117.

d=1

For $d = 1$, consider the graph $K_{1,n-1}$ and add a single edge inside the part of size $n - 1$. This graph can be oriented easily so that the minimum outdegree is 1 and

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \leq 1$$

for every partition (V_1, V_2) of such a graph. Hence,

$$c_1 = 0.$$

Lee-Loh-Sudakov Conjecture

Lee, Loh and Sudakov initiated the study of this problem and conjectured that

Conjecture

Let d be an integer satisfying $d \geq 2$. Every digraph D with m arcs and minimum outdegree at least d admits a bipartition (V_1, V_2) for which

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geq \left(\frac{d-1}{2(2d-1)} + o(1)\right)m.$$

C. Lee, P. Loh, B. Sudakov, Judicious partitions of directed graphs, Random Struct. Alg. 48 (2016) 147–170.

Extremal graph

The bound in Conjecture 3 is asymptotically best possible by considering the following extremal graph:

First orient the arcs of the complete graph K_{2d-1} along an Eulerian circuit.

Then consider the directed graph where we take k vertex disjoint copies of K_{2d-1} oriented as above, and a single vertex disjoint copy of K_{2d+1} oriented in a similar manner.

Fix a vertex v_0 of K_{2d+1}, and add arcs so that all the vertices belonging to the copies of K_{2d-1} are in-neighbors of v_0.

Continued

Clearly, the resulting graph has minimum outdegree d, minimum degree $2d - 1$, minimum semidegree $d - 1$.

Progress

In the same paper, they confirmed the conjecture when $d = 2, 3$.

On the other hand, they noted that the methods they used turn out to be too limited in strength to cover the cases $d \geq 4$.

C. Lee, P. Loh, B. Sudakov, Judicious partitions of directed graphs, Random Struct. Alg. 48 (2016) 147–170.

Dense case

For digraphs with bounded maximum degree or large number of arcs, we can get a better bound.

Theorem (H., Wu, Yan, EJC, 2017)

Let D be a digraph with n vertices and m arcs. For every $\epsilon > 0$, if $m \geq 16n/\epsilon^2$ or the maximum degree Δ of D is at most $\epsilon^2 m/8$, then D admits a bisection (V_1, V_2) such that

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geq \frac{1}{4}m - \epsilon m.$$

Standard approach

A standard approach to find a “good" bipartition of a digraph D is to first partition $V(D)$ into X, Y, where X consists of certain high degree vertices.

Then, partition X into X_1 and X_2 with certain property;

Finally, apply a randomized algorithm to distribute the vertices in Y.

B. Bollobás, A.D. Scott, Judicious partitions of hypergraphs, J. Combin. Theory Ser. A 78 (1997) 15–31.

Difficulty

The main step in this approach is to deal with the arcs between X and Y.

The condition that “the minimum outdegree of D is at least d" yields that the number of arcs from Y to X can be bounded.

However, if we bound the number of arcs from X to Y, we need to know the indegree of vertices in Y.

Semidegree

Our first result shows that Conjecture 3 holds under the natural (additional) assumption that the minimum indegree of D is at least d.

Theorem (H., Ma, Yu, Zhang, 2019, SCM)

Let $d \geq 2$ be an integer. Every digraph D with m arcs and minimum semidegree at least d admits a bipartition (V_1, V_2) for which

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geq \left(\frac{d-1}{2(2d-1)} + o(1)\right)m.$$

Oriented graph

Note that an oriented graph is an orientation of a simple graph. If we focus on oriented graphs with minimum semidegree d, then we can give a better bound.

Theorem (H., Wu, JCTB, 2018)

Let d be an integer with $d \geq 21$. Every oriented graph D with m arcs and minimum semidegree d admits a bipartition (V_1, V_2) such that

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geq \left(\frac{d}{2(2d+1)} + o(1)\right)m.$$

Special digraphs

Let $\overrightarrow{K_{d,2}}$ be the digraph obtained by orient each edge of a bipartite graph $K_{2,d}$ from the part of size d to the other part.

Theorem (H., L., Wu, 2019+)

Let $d \geq 4$ be an integer and D is a digraph with m arcs and minimum outdegree at least d. If D does not contain $\overrightarrow{K_{d,2}}$, then there is a bipartition (V_1, V_2) such that

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geq \left(\frac{d-1}{2(2d-1)} + o(1)\right)m.$$

Underlying graph

The underlying graph of a digraph is obtained by ignoring arc orientations and removing redundant parallel arcs when arcs in both directions appear between pairs of vertices.

Without 4-cycles

For digraphs whose underlying graph does not have 4-cycles, we have

Theorem (H., Wu, Yan, EJC, 2017)

Let D be a digraph with m arcs and minimum outdegree at least 2, and let G be its underlying graph. If G does not contain 4-cycles, then D admits a bisection (V_1, V_2) such that

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geq \left(\frac{1}{4} + o(1)\right)m.$$

Conclusion (1)

The Max-Cut problem is a fundamental discrete optimization problem and have been studied widely. We give the lower bound on Max-Cut of graphs without specific structure.

There are litter results on Max-Bisections of graphs. Even for C_4-free graphs G, we do not know the maximum constant c such that the following holds:

If G has m edges and minimum degree $\delta \geq 2$, then G has a bisection of size at least

$$m/2 + \Omega(m^c).$$

Conclusion (2)

We conclude our discussion with the following question:

Question (H., Wu, JCTB, 2018)

Is it true that for every integer $d \geq 1$, every digraph D with m arcs and minimum semidegree d admits a bipartition $V(D) = V_1 \cup V_2$ such that

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geq \left(\frac{d}{2(2d+1)} + o(1)\right) m?$$

As evidence, we have showed that it is true for digraphs with minimum semidegree at most 3.

Thank you for your attention!