Semidefinite programming bounds for spherical three-distance sets

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National Central University
Joint work with Feng-Yuan Liu (劉豐源)

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Outline

- Introduction & history
- Harmonic absolute bound
- Linear programming (LP)
- Semidefinite programming (SDP)
- Discrete sampling points with Nozaki theorem
- Rigorous proof with sum of squares method (SOS)

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- Introduction & history

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- Harmonic absolute bound
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- Semidefinite programming (SDP)

experiment technique

- <u>Discrete sampling</u> points with Nozaki theorem
- Rigorous proof with <u>sum of squares</u> method (SOS)

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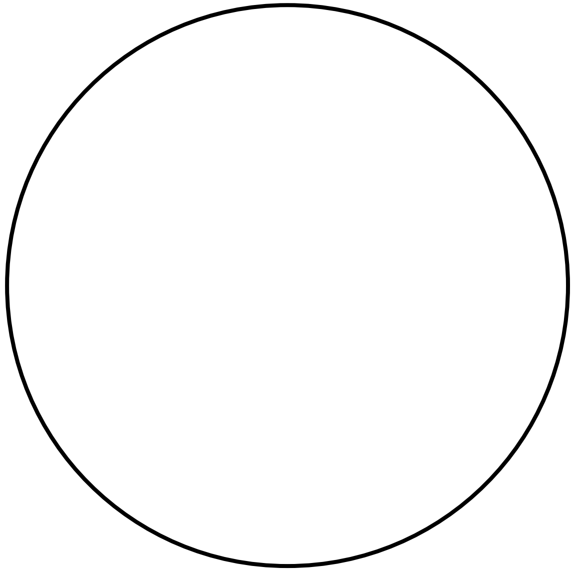
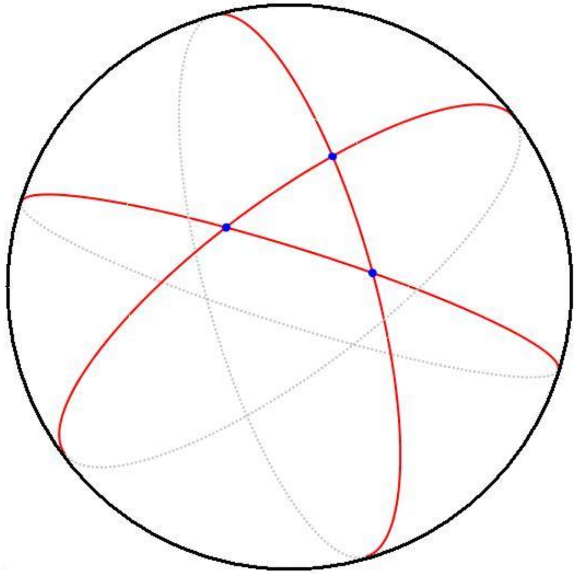
spherical

- $S^{n-1} := \{x \in \mathbb{R}^n : \langle x, x \rangle = 1\}$

Ex. $n = 2$
S^1 : Circle



Ex. $n = 3$
S^2 : Ball



s-distance set

$$\boldsymbol{X} = \{x_1, x_2, x_3, \dots, x_k\}, X \subset S^{n-1}$$

$$\left\|x_i - x_j\right\|_2 = \{l_1, l_2, l_3, \dots, l_{\boldsymbol{s}}\}\ \forall i \neq j$$

Max s-distance set

$\boldsymbol{X} = \{x_1, x_2, x_3, \dots, x_k\}, X \subset S^{n-1}$

Object: $\boldsymbol{max}\ |\boldsymbol{X}|$

Subject to: $\|x_i - x_j\|_2 = \{l_1, l_2, l_3, \dots, l_{\boldsymbol{s}}\}\ \forall i \neq j$

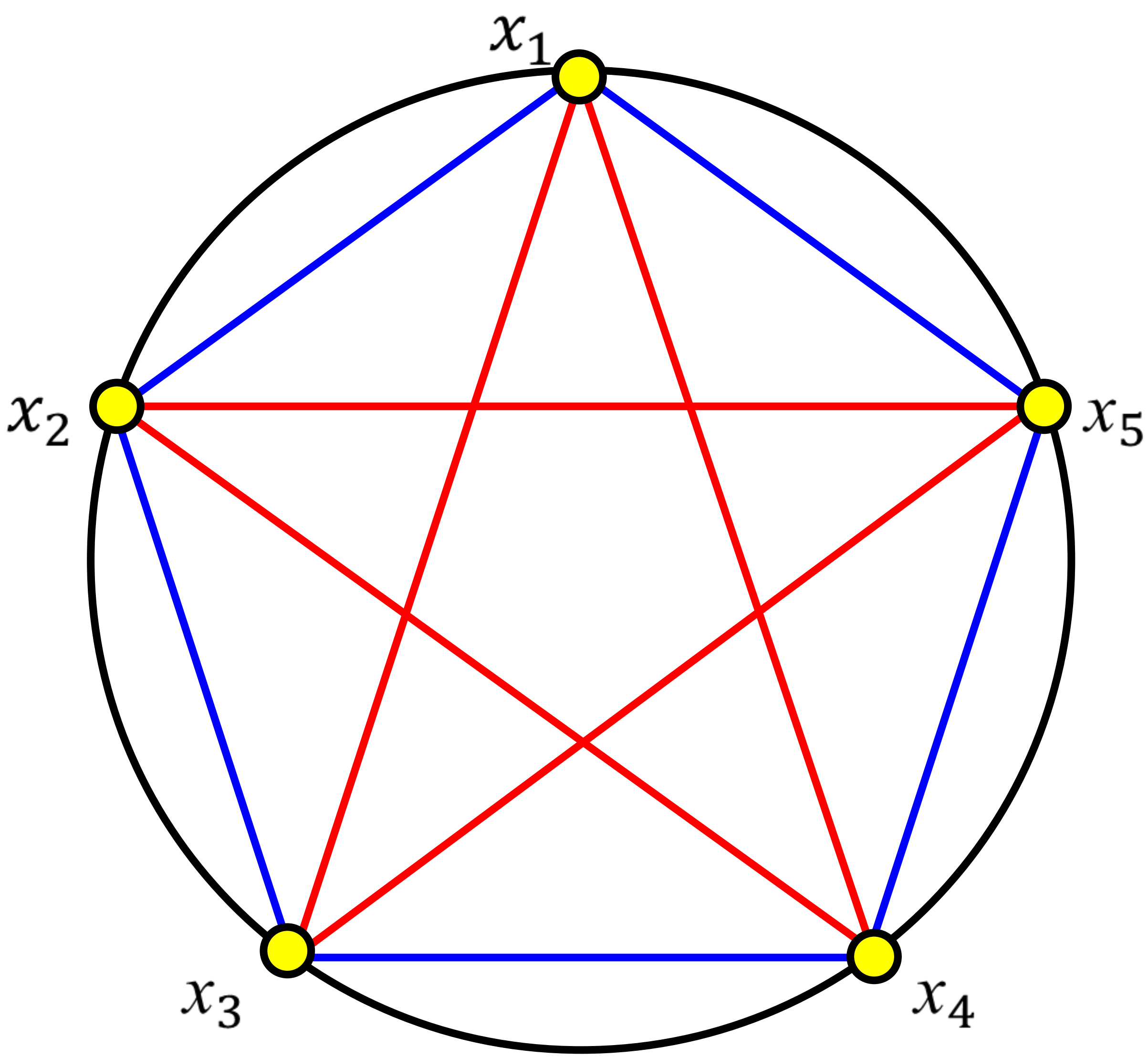
Max 2-distance set

$$X = \{x_1, x_2, x_3, \dots, x_k\}, X \subset S^{n-1}$$

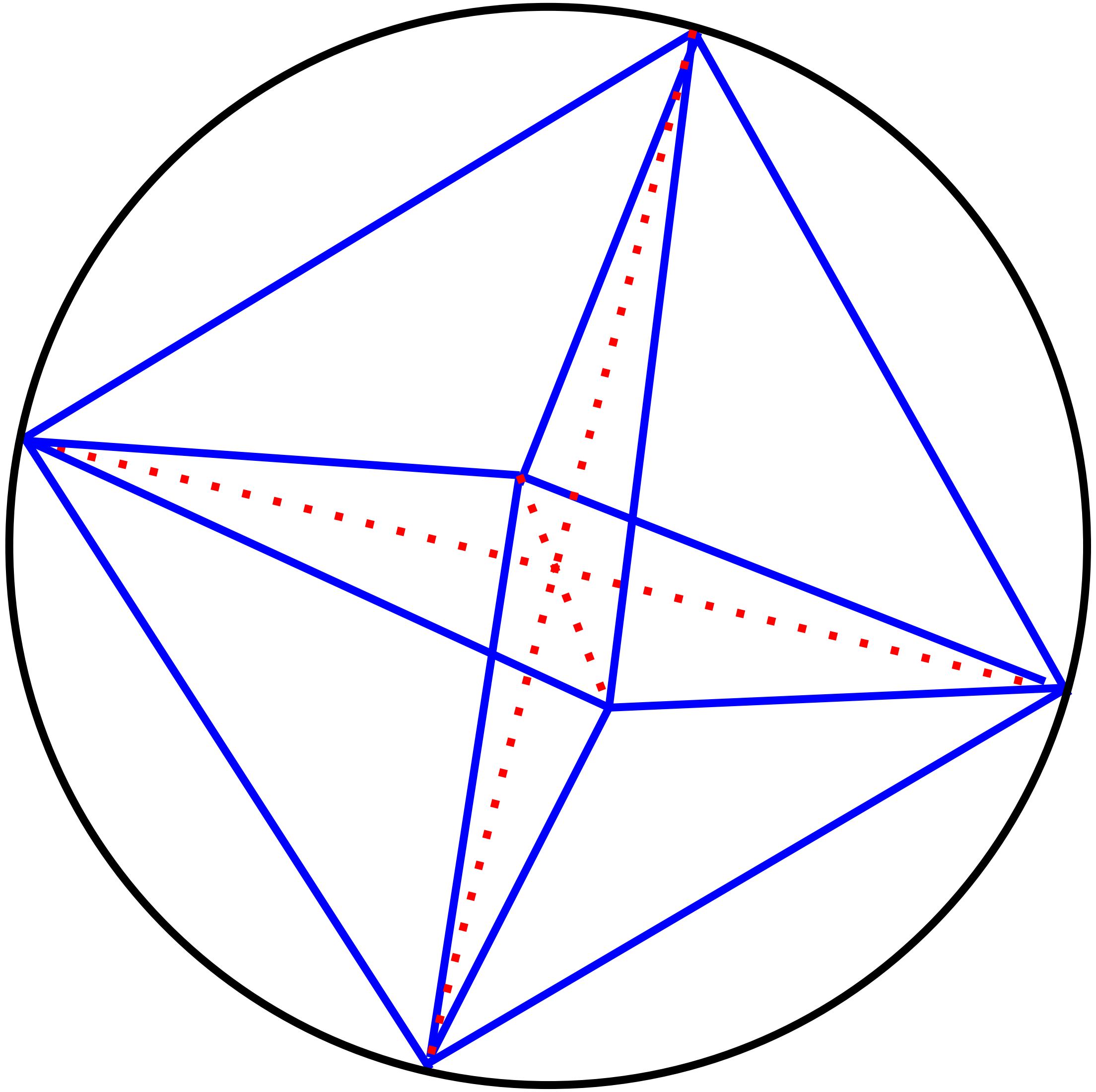
Object: $\max |X|$

Subject to: $\|x_i - x_j\|_2 = \{l_1, l_2\} \ \forall i \neq j$

Max spherical 2-distance set in R^2



Max spherical 2-distance set in R^3



Max spherical 2-distance set in R^n

(n)	bound	
2	5	(Pentagon)
3	6	(Octahedron)
4	10	(Petersen graph)
5	16	
6	27	

Max spherical 2-distance set in R^n

(n)	bound	
2	5	(Pentagon)
3	6	(Octahedron)
4	10	(Petersen graph)
5	16	
6	27	

- $6 < n \leq 22$, $23 < n < 40$: O. R. Musin, 2008, LP
- $n = 23$, $40 \leq n \leq 93$, (except $n = 46, 78$): A. Barg & W. H. Yu, 2013, SDP
- $n \leq 417$: W. H. Yu, 2016
- for all n, except $n=(2k+1)^2-3$: A. Glazyrin, W. H. Yu, 2018 (Adv. in math)

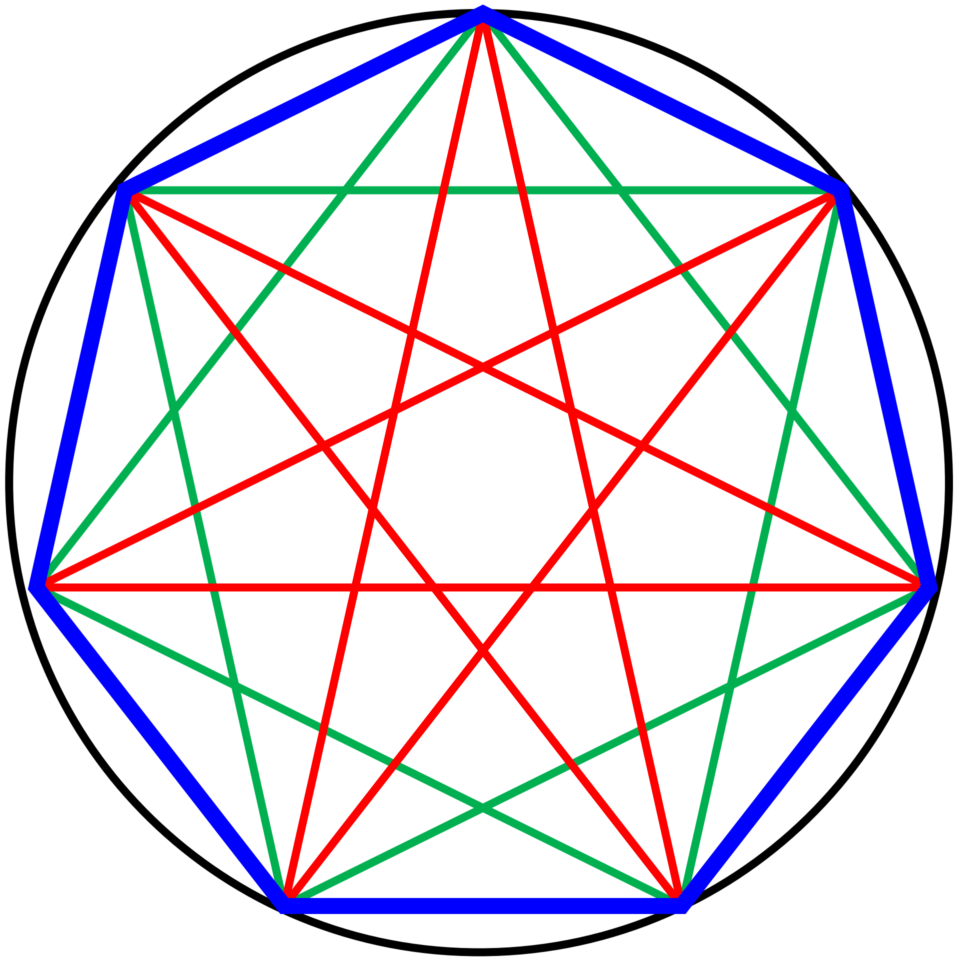
Max 3-distance set

$$\boldsymbol{X} = \{x_1, x_2, x_3, \ldots, x_k\}, X \subset S^{n-1}$$

Object: $\boldsymbol{max}\ |\boldsymbol{X}|$

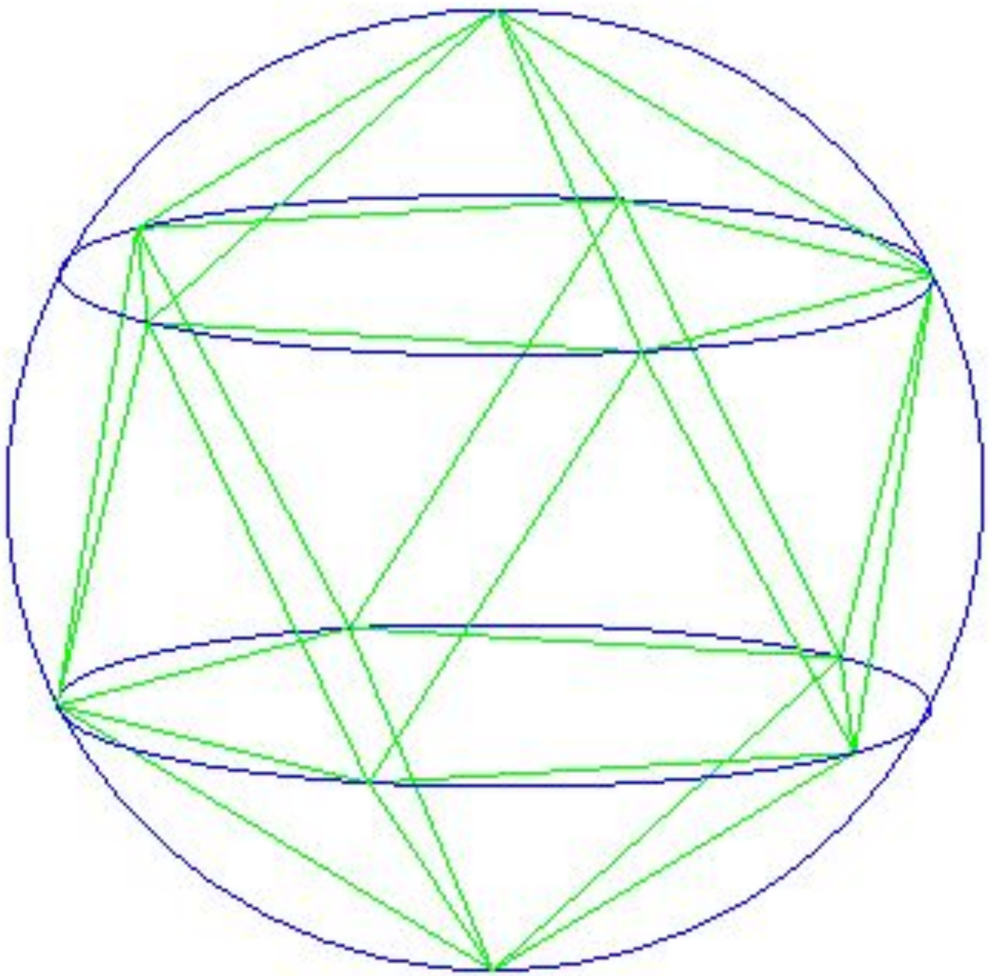
Subject to: $\|x_i - x_j\|_2 = \{\boldsymbol{l_1}, \boldsymbol{l_2}, \boldsymbol{l_3}\}\ \forall i \neq j$

Max spherical 3-distance set in R^2



Regular Heptagon

Max spherical *3*-distance set in R^3

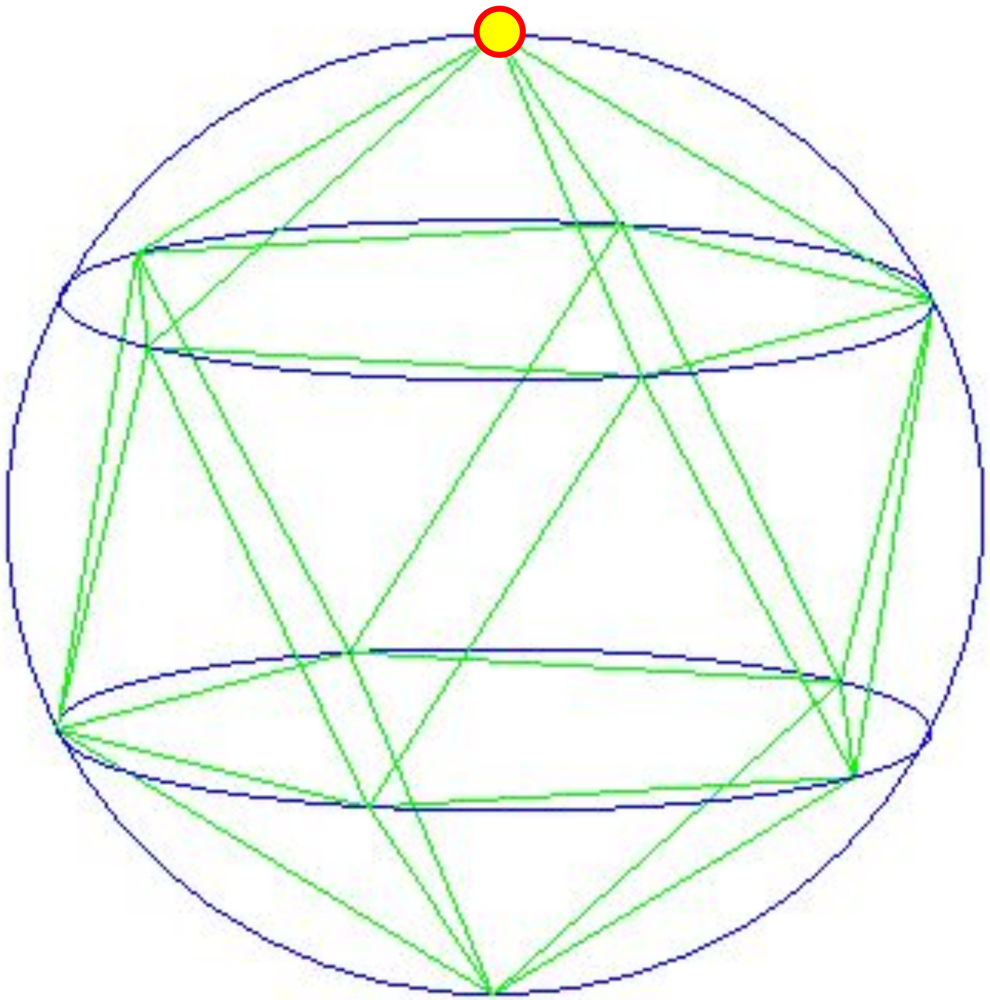


$$\|x_i - x_j\|_2 = \begin{cases} a \\ b \\ c \end{cases}$$

$$i \neq j$$

Regular icosahedron

Max spherical 3-distance set in R^3



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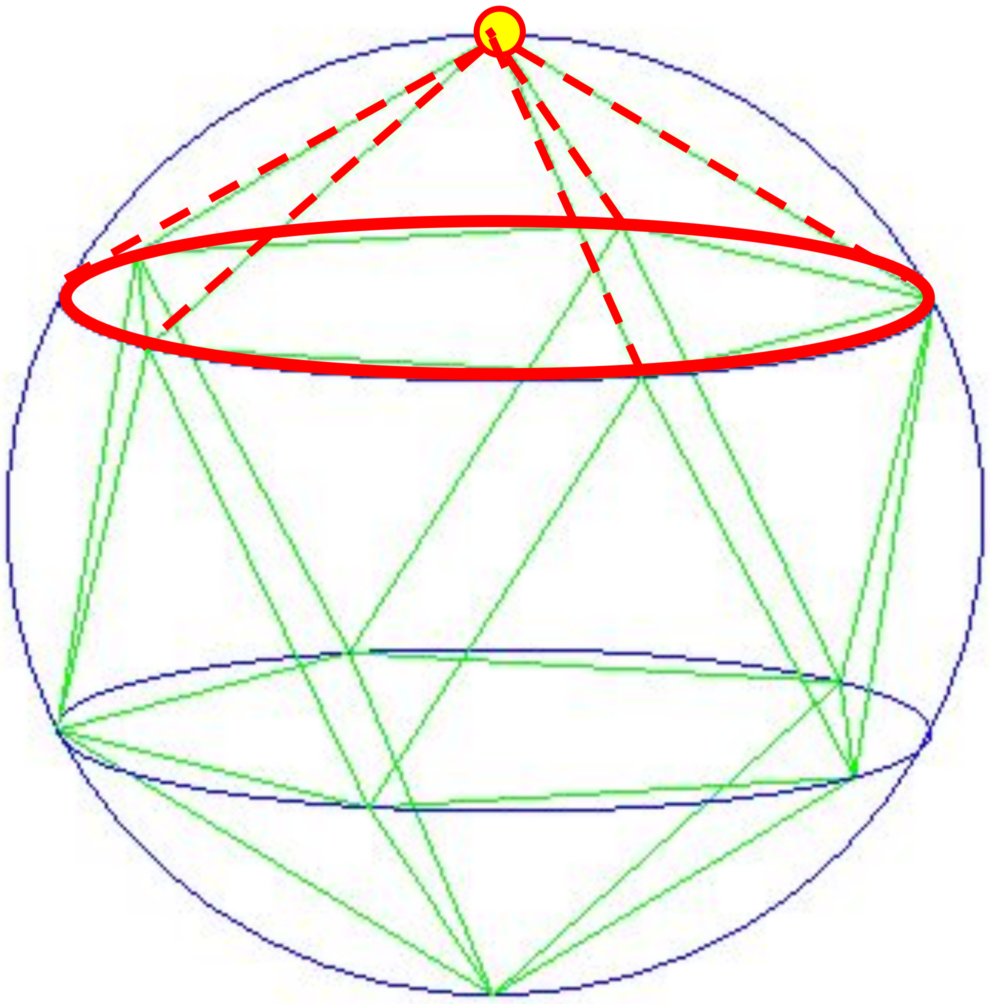

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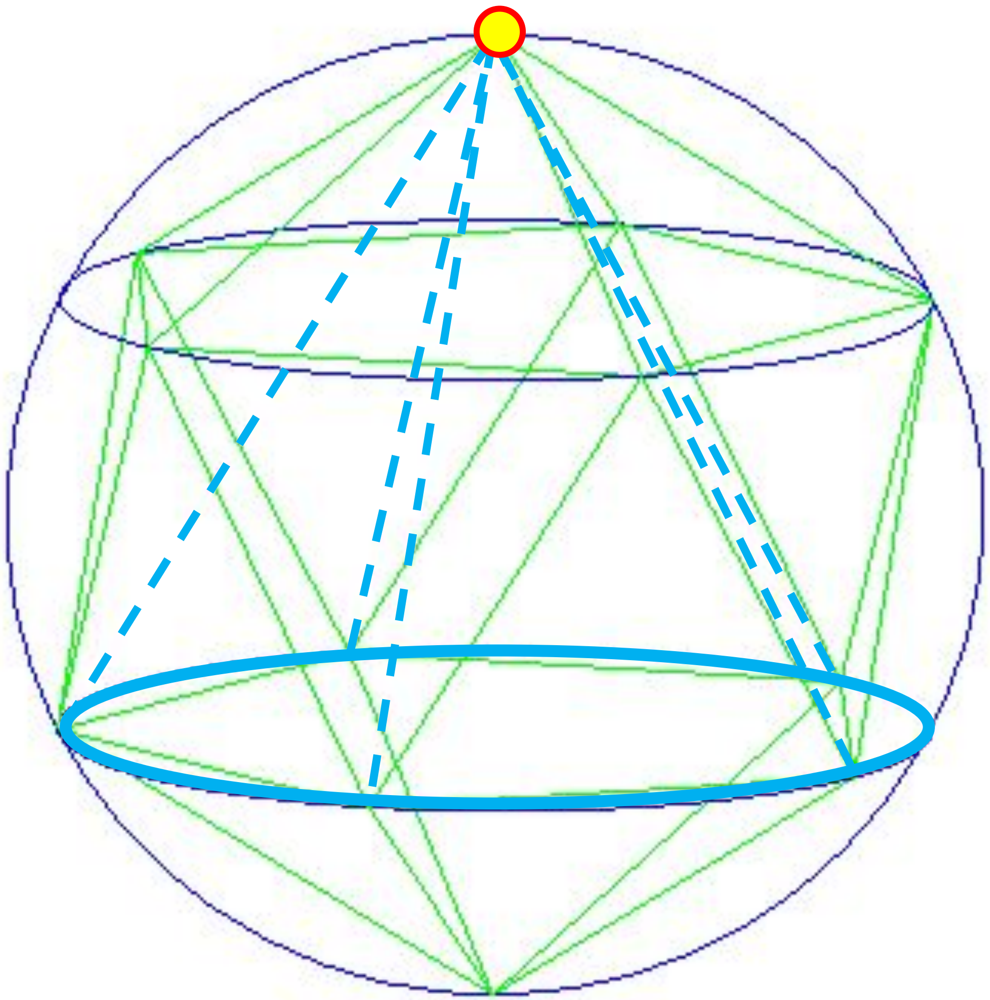

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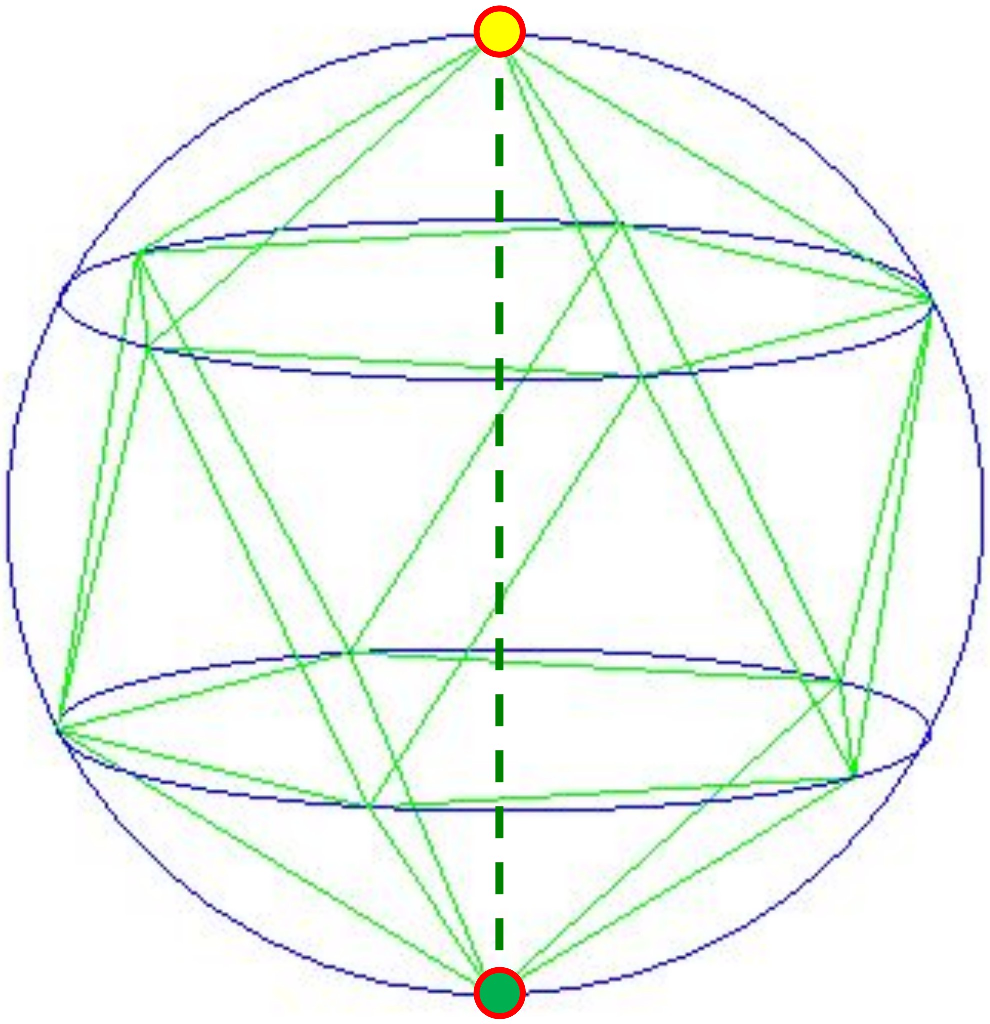

$$\|x_i - x_j\|_2 = \begin{cases} a \\ b \\ c \end{cases}$$

$$i \neq j$$

Regular icosahedron

Max spherical 3-distance set in R^3

- Regular icosahedron is the unique max spherical 3-dis set in R^3

Uniqueness of maximum three-distance sets in the three-dimensional Euclidean space

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Abstract

A subset X in the d-dimensional Euclidean space is called a k-distance set if there are exactly k distances between two distinct points in X. Einhorn and Schoenberg conjectured that the vertices of the regular icosahedron is the only 12-point three-distance set in $\mathbb{R}^3$ up to isomorphism. In this paper, we prove the uniqueness of 12-point three-distance sets in $\mathbb{R}^3$.

Max Spherical 3-distance set in R^n

(n)	bound	
2	7	(Heptagon)
3	12	(Icosahedron), M. Shinohara, 2013
4	13	F. Szöllősi & P. R. J. Östergård, 2018

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5	≤ 39	**(LP)** Musin & Nozaki, 2010
6	≤ 56	**(LP)** Musin & Nozaki, 2010
7	≤ 91	**(LP)** Musin & Nozaki, 2010
8	120	(Subset of E_8 root system), **(LP)** Musin & Nozaki, 2010
22	2025	(Subset of Leech lattice), **(LP)** Musin & Nozaki, 2010
23	≤ 2301	**(LP)** Musin & Nozaki, 2010

Max Spherical 3-distance set in R^n

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7	≤ 91 **≤ 84**	(LP) Musin & Nozaki, 2010 **(SDP) F. Y. Liu & W. H. Yu, 2019+**
8	120	(Subset of E_8 root system), (LP) Musin & Nozaki, 2010
22	2025	(Subset of Leech lattice), (LP) Musin & Nozaki, 2010
23	≤ 2301 **2300**	(LP) Musin & Nozaki, 2010 **(A half of Tight spherical 7-design), (SDP) F. Y. Liu & W. H. Yu, 2019+**

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Upper bound of spherical 3-distance set

Harmonic absolute bound

- Proved by Delsarte
- Nozaki improved this bound

Delsarte’s linear programming bound

Upper bound of spherical *3*-distance set

Harmonic absolute bound

- Proved by Delsarte
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Delsarte's linear programming bound

+ Semidefinite programming bound

Max *3*-distance set

$$\boldsymbol{X} = \{x_1, x_2, x_3, \ldots, x_k\}, X \subset S^{n-1}$$

Object: $\boldsymbol{max}\ |\boldsymbol{X}|$

Subject to: $\|x_i - x_j\|_2 = \{\boldsymbol{l_1}, \boldsymbol{l_2}, \boldsymbol{l_3}\}$

$$\Updownarrow$$

$$\langle x_i, x_j \rangle = \{\boldsymbol{d_1}, \boldsymbol{d_2}, \boldsymbol{d_3}\}$$

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$\Updownarrow$ **Law of cosines**

$$\langle x_i, x_j \rangle = \{\boldsymbol{d_1}, \boldsymbol{d_2}, \boldsymbol{d_3}\}$$

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Gegenbauer polynomials

Definition *(Gegenbauer polynomials)*

Denote the Gegenbauer polynomials of degree k in $\mathbb{R}^n$ by $G_k^n(t)$. They are defined with the following recursive relationship:

$$G_0^n(t) \equiv 1, G_1^n(t) = t$$

$$G_k^n(t) = \frac{(2k+n-4)\, t\, G_{k-1}^n(t) - (k-1) G_{k-2}^n(t)}{k+n-3}, k \geq 2$$

Harmonic Absolute Bound

- **Theorem (Nozaki)**
  Let X be an 3-distance set in S^{n-1} with $D(X)=\{d_1,d_2,d_3\}$.
  Consider the polynomial $f(x) = (d_1 - x)(d_2 - x)(d_3 - x)$

Harmonic Absolute Bound

- **Theorem (Nozaki)**
  Let X be an 3-distance set in S^{n-1} with $D(X)=\{d_1,d_2,d_3\}$.
  Consider the polynomial $f(x)=(d_1-x)(d_2-x)(d_3-x)$
  and suppose that its expansion in the basis $\{G_k^n\}$ has the form $f(x)=\sum_{k=0}^{3} f_k^n G_k^n(x)$.

Gegenbauer polynomials

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  Gegenbauer polynomials
  Then $|X| \leq \sum_{k: f_k^n > 0} h_k^n$.
  Harmonic bound

Harmonic Absolute Bound

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  Gegenbauer polynomials

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  Harmonic bound

- $h_k^n := \dim\left(\mathrm{Harm}_k(\mathbb{R}^n)\right) = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$

  dimension of linear space on all real harmonic homogeneous polynomials of degree k in $\mathbb{R}^n$

Example: $n = 23$, $(d_1, d_2, d_3) = (-1/3, 0, 1/3)$

$$\begin{cases} f(x) = (d_1 - x)(d_2 - x)(d_3 - x). \\ f(x) = \sum_{k=0}^{3} f_k^n G_k^n(x). \end{cases}$$

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$$G_0^n(x) = 1$$

$$G_1^n(x) = x$$

$$G_2^n(x) = \frac{nx^2 - 1}{n - 1}$$

$$G_3^n(x) = \frac{x}{n - 1}(nx^2 + 2x^2 - 3)$$

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$$f_0^n = -d_1 d_2 d_3 - \frac{d_1 + d_2 + d_3}{n}$$

$$f_1^n = d_1 d_2 + d_1 d_3 + d_2 d_3 + \frac{3}{n+2}$$

$$f_2^n = \frac{1-n}{n}(d_1 + d_2 + d_3)$$

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$$f_0^n = -d_1 d_2 d_3 - \frac{d_1 + d_2 + d_3}{n} \quad \leq 0$$

$$f_1^n = d_1 d_2 + d_1 d_3 + d_2 d_3 + \frac{3}{n+2} \quad > 0$$

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$$|X| \leq \sum_{k: f_k^n > 0} h_k^n$$

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$$|X| \leq \sum_{k: f_k^n > 0} h_k^n$$

$$= h_1^n + h_3^n$$

$$= 23 + 2277$$

$$= \mathbf{2300}$$

$$h_k^n = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$$

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Linear programming (LP)

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \\
& \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m \\
& x_j \ge 0 \quad \text{for } j = 1, 2, \ldots, n\,.
\end{aligned}
$$

Upper bound of spherical 3-distance set (LP)

Theorem *(Delsarte's inequality)*

For any finite set of points $X \subset S^{n-1}$

$$\sum_{(x,y)\in X^2} G_k^n(x \cdot y) \geq 0, k = 1, 2, 3, \cdots$$

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Theorem *(Delsarte's linear programming bound)*

Let $X \in S^{n-1}$ *be a finite set and assume that for any* $x, y \in X$, $\tau(x, y) \in \{d_1, d_2, d_3\}$. *Then the cardinality of* $|X|$ *is bounded above by the solution of the following linear programming problem:*

$$\begin{aligned} &\textit{maximize} && 1 + x_1 + x_2 + x_3 \\ &\textit{subject to} && 1 + x_1 G_k^n(d_1) + x_2 G_k^n(d_2) + x_3 G_k^n(d_3) \geq 0, k = 1, 2, 3, \ldots \\ & && x_j \geq 0, j = 1, 2, 3. \end{aligned}$$

Semidefinite Programming (SDP)

$$\text{minimize} \quad c^T x \qquad (x \in \mathbb{R}^m)$$

$$\text{subject to} \quad F(x) \succcurlyeq 0$$

where

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i$$

and vector $c \in \mathbb{R}^m$. $F_0, \cdots, F_m$ are symmetric matrices in $\mathbb{R}^{n \times n}$. The inequality sign in $F(x) \succcurlyeq 0$ means that $F(x)$ is positive semidefinite, i.e.,

$$z^T F z \geq 0, \forall z \in \mathbb{R}^n$$

Upper bound (SDP)

Schoenberg (1942)

$$\sum_{(x,y,z)\in X^3} S_k^n(x\cdot y, x\cdot z, y\cdot z) \succcurlyeq 0$$

$$\begin{aligned}
&\textit{maximize} \quad 1+\frac{1}{3}(x_1+x_2+x_3)\\
&\textit{subject to} \quad \begin{pmatrix}1&0\\0&0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0&1\\1&1\end{pmatrix}(x_1+x_2+x_3)+\begin{pmatrix}0&0\\0&1\end{pmatrix}\sum_{i=4}^{13}x_i \succcurlyeq 0\\
&3+x_1G_k^n(d_1)+x_2G_k^n(d_2)+x_3G_k^n(d_3)\geq 0,\ k=1,2,\cdots,p_{LP}\\
&S_k^n(1,1,1)+x_1S_k^n(d_1,d_1,1)+x_2S_k^n(d_2,d_2,1)+x_3S_k^n(d_3,d_3,1)\\
&+x_4S_k^n(d_1,d_1,d_1)+x_5S_k^n(d_2,d_2,d_2)+x_6S_k^n(d_3,d_3,d_3)\\
&+x_7S_k^n(d_1,d_1,d_2)+x_8S_k^n(d_1,d_1,d_3)+x_9S_k^n(d_2,d_2,d_1)\\
&+x_{10}S_k^n(d_2,d_2,d_3)+x_{11}S_k^n(d_3,d_3,d_1)+x_{12}S_k^n(d_3,d_3,d_2)\\
&+x_{13}S_k^n(d_1,d_2,d_3)\succcurlyeq 0,\ k=0,1,2,\cdots,p_{SDP}\\
&x_j\geq 0,\ j=1,2,\cdots,13
\end{aligned}$$

Upper bound (SDP)

Schoenberg (1942)

$$\sum_{(x,y,z)\in X^3} S_k^n(x\cdot y, x\cdot z, y\cdot z) \succcurlyeq 0$$

$$\text{maximize} \quad 1+\frac{1}{3}(x_1+x_2+x_3)$$

$$\text{subject to} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(x_1+x_2+x_3) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\sum_{i=4}^{13} x_i \succcurlyeq 0$$

LP $3 + x_1 G_k^n(d_1) + x_2 G_k^n(d_2) + x_3 G_k^n(d_3) \geq 0, \ k = 1, 2, \cdots, p_{LP}$

$$S_k^n(1,1,1) + x_1 S_k^n(d_1,d_1,1) + x_2 S_k^n(d_2,d_2,1) + x_3 S_k^n(d_3,d_3,1)$$

$$+ x_4 S_k^n(d_1,d_1,d_1) + x_5 S_k^n(d_2,d_2,d_2) + x_6 S_k^n(d_3,d_3,d_3)$$

$$+ x_7 S_k^n(d_1,d_1,d_2) + x_8 S_k^n(d_1,d_1,d_3) + x_9 S_k^n(d_2,d_2,d_1)$$

$$+ x_{10} S_k^n(d_2,d_2,d_3) + x_{11} S_k^n(d_3,d_3,d_1) + x_{12} S_k^n(d_3,d_3,d_2)$$

$$+ x_{13} S_k^n(d_1,d_2,d_3) \succcurlyeq 0, \ k = 0, 1, 2, \cdots, p_{SDP}$$

$$x_j \geq 0, \ j = 1, 2, \cdots, 13$$

Upper bound (SDP)

Schoenberg (1942)

$$\sum_{(x,y,z)\in X^3} S_k^n(x\cdot y, x\cdot z, y\cdot z) \succcurlyeq 0$$

$$\textit{maximize} \quad 1+\frac{1}{3}(x_1+x_2+x_3)$$

$$\textit{subject to} \quad \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}(x_1+x_2+x_3) + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\sum_{i=4}^{13} x_i \succcurlyeq 0$$

LP

$$3 + x_1 G_k^n(d_1) + x_2 G_k^n(d_2) + x_3 G_k^n(d_3) \geq 0,\ k = 1, 2, \cdots, p_{LP}$$

Schoenberg

$$\begin{aligned}
& S_k^n(1,1,1) + x_1 S_k^n(d_1,d_1,1) + x_2 S_k^n(d_2,d_2,1) + x_3 S_k^n(d_3,d_3,1) \\
& + x_4 S_k^n(d_1,d_1,d_1) + x_5 S_k^n(d_2,d_2,d_2) + x_6 S_k^n(d_3,d_3,d_3) \\
& + x_7 S_k^n(d_1,d_1,d_2) + x_8 S_k^n(d_1,d_1,d_3) + x_9 S_k^n(d_2,d_2,d_1) \\
& + x_{10} S_k^n(d_2,d_2,d_3) + x_{11} S_k^n(d_3,d_3,d_1) + x_{12} S_k^n(d_3,d_3,d_2) \\
& + x_{13} S_k^n(d_1,d_2,d_3) \succcurlyeq 0,\ k = 0, 1, 2, \cdots, p_{SDP}
\end{aligned}$$

$$x_j \geq 0,\ j = 1, 2, \cdots, 13$$

Upper bound (SDP)

Schoenberg (1942)

$$\sum_{(x,y,z)\in X^3} S_k^n(x\cdot y, x\cdot z, y\cdot z) \succcurlyeq 0$$

maximize $1 + \frac{1}{3}(x_1 + x_2 + x_3)$

subject to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(x_1 + x_2 + x_3) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\sum_{i=4}^{13} x_i \succcurlyeq 0$

Bachoc & Vallentin (kissing number)

LP $3 + x_1 G_k^n(d_1) + x_2 G_k^n(d_2) + x_3 G_k^n(d_3) \geq 0,\ k = 1, 2, \cdots, p_{LP}$

Schoenberg

$$S_k^n(1,1,1) + x_1 S_k^n(d_1,d_1,1) + x_2 S_k^n(d_2,d_2,1) + x_3 S_k^n(d_3,d_3,1)$$
$$+ x_4 S_k^n(d_1,d_1,d_1) + x_5 S_k^n(d_2,d_2,d_2) + x_6 S_k^n(d_3,d_3,d_3)$$
$$+ x_7 S_k^n(d_1,d_1,d_2) + x_8 S_k^n(d_1,d_1,d_3) + x_9 S_k^n(d_2,d_2,d_1)$$
$$+ x_{10} S_k^n(d_2,d_2,d_3) + x_{11} S_k^n(d_3,d_3,d_1) + x_{12} S_k^n(d_3,d_3,d_2)$$
$$+ x_{13} S_k^n(d_1,d_2,d_3) \succcurlyeq 0,\ k = 0, 1, 2, \cdots, p_{SDP}$$

$x_j \geq 0,\ j = 1, 2, \cdots, 13$

Schoenberg (1942)

$$\sum_{(x,y,z)\in X^3} S_k^n(x\cdot y, x\cdot z, y\cdot z) \succcurlyeq 0$$

$$S_k^n(u,v,t) = \frac{1}{6}\sum_{\sigma\in S_3} Y_k^n(\sigma(u,v,t))$$

$$(Y_k^n(u,v,t))_{ij} = u^i v^j((1-u^2)(1-v^2))^{\frac{k}{2}} G_k^{n-1}(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}})$$

$$\sum_{(x,y)\in C^2} G_k^n(x \cdot y) \geq 0$$ LP

$$\sum_{(x,y,z)\in C^3} S_k^n(x \cdot y, x \cdot z, y \cdot z) \succcurlyeq 0$$ SDP

$$\sum_{(x,y)\in C^2} G_k^n(x\cdot y) \geq 0$$ LP

$$\sum_{(x,y,z)\in C^3} S_k^n(x\cdot y, x\cdot z, y\cdot z) \succcurlyeq 0$$ SDP

$D(x) = \{d_1, d_2, d_3\}$

$$\sum_{(x,y)\in C^2} G_k^n(x \cdot y) \geq 0$$ LP

$$\sum_{(x,y,z)\in C^3} S_k^n(x \cdot y, x \cdot z, y \cdot z) \succcurlyeq 0$$ SDP

$D(x) = \{d_1, d_2, d_3\} \Rightarrow D(x) = \{F_1(\mathbf{d_3}), F_2(\mathbf{d_3}), \mathbf{d_3}\}$

Outline

- Introduction & history
- Harmonic absolute bound
- Linear programming (LP)
- Semidefinite programming (SDP)
- **Discrete sampling points with Nozaki theorem**

  Generalization of Larman-Rogers-Seidel's theorem
- Rigorous proof with sum of squares method (SOS)

Generalization of Larmen-Rogers-Seidel's theorem

Definition $K_i := \prod_{j\neq i} \dfrac{d_j - 1}{d_j - d_i}$

$$K_1 = \frac{d_2 - 1}{d_2 - d_1} \cdot \frac{d_3 - 1}{d_3 - d_1}, \quad K_2 = \frac{d_1 - 1}{d_1 - d_2} \cdot \frac{d_3 - 1}{d_3 - d_2}, \quad K_3 = \frac{d_1 - 1}{d_1 - d_3} \cdot \frac{d_2 - 1}{d_2 - d_3}$$

Theorem (Nozaki).
Let X be an *3*-distance set in S^{n-1} with $D(X)=\{d_1,d_2,d_3\}$
If $|X| \geq 2N(S^{n-1}, 3)$
then ***K_i* is an integer** for each i=1,2,3.
Also $|K_i| \leq \lfloor 1/2+\sqrt{N(S^{n-1},3)^2/(2N(S^{n-1},3)-2+1/4)} \rfloor$.

Generalization of Larmen-Rogers-Seidel's theorem

```
>> Yuan_Ki(7)

ans =

     1     1     1     2     2     3
    -4    -3    -2    -4    -3    -4
     4     3     2     3     2     2
```

Theorem (Nozaki).
Let X be an *3*-distance set in S^{n-1} with $D(X)=\{d_1,d_2,d_3\}$
If $|X| \geq 2N(S^{n-1},3)$
then K_i **is an integer** for each i=1,2,3.
Also $|K_i| \leq \lfloor 1/2+\sqrt{N(S^{n-1},3)^2/(2N(S^{n-1},3)-2+1/4)} \rfloor$.

Generalization of Larmen-Rogers-Seidel's theorem

Theorem. (Musin & Nozaki)

$\sum_{i=1}^{3} d_i^{\,j} K_i = 1$ (j=0,1,2)

Generalization of Larmen-Rogers-Seidel's theorem

Theorem. (Musin & Nozaki)

$\sum_{i=1}^{3} d_i{}^j K_i = 1$ (j=0,1,2)

- $K_1 + K_2 + K_3 = 1$
- $d_1K_1 + d_2K_2 + d_3K_3 = 1$
- $d_1{}^2K_1 + d_2{}^2K_2 + d_3{}^2K_3 = 1$

Generalization of Larmen-Rogers-Seidel's theorem

Theorem. (Musin & Nozaki)

$\sum_{i=1}^{3} d_i^{\,j} K_i = 1$ (j=0,1,2)

- $K_1 + K_2 + K_3 = 1$
- $d_1 K_1 + d_2 K_2 + d_3 K_3 = 1$
- $d_1^2 K_1 + d_2^2 K_2 + d_3^2 K_3 = 1$
- $d_1 < d_2$

Generalization of Larmen-Rogers-Seidel's theorem

Theorem. (Musin & Nozaki)

$\sum_{i=1}^{3} d_i^{\,j} K_i = 1$ (j=0,1,2)

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- $d_1^2K_1 + d_2^2K_2 + d_3^2K_3 = 1$
- $d_1 < d_2$

$$d_1 = \frac{K_1 - d_3K_1K_3 - (d_3 - 1)\sqrt{-K_1K_2K_3}}{K_1(K_1+K_2)}$$

$$d_2 = \frac{K_2 - d_3K_2K_3 + (d_3 - 1)\sqrt{-K_1K_2K_3}}{K_2(K_1+K_2)}$$

Generalization of Larmen-Rogers-Seidel's theorem

Theorem. (Musin & Nozaki)

$\sum_{i=1}^{3} d_i^{\,j} K_i = 1$ (j=0,1,2)

- $K_1 + K_2 + K_3 = 1$
- $d_1K_1 + d_2K_2 + d_3K_3 = 1$
- $d_1^2K_1 + d_2^2K_2 + d_3^2K_3 = 1$
- $d_1 < d_2$

Given K_1, K_2, K_3

d_1, d_2 are function of d_3

$$d_1 = \frac{K_1 - d_3K_1K_3 - (d_3 - 1)\sqrt{-K_1K_2K_3}}{K_1(K_1+K_2)}$$

$$d_2 = \frac{K_2 - d_3K_2K_3 + (d_3 - 1)\sqrt{-K_1K_2K_3}}{K_2(K_1+K_2)}$$

Upper bound of spherical 3-distance set (LP)

- Object: $\boldsymbol{max}(\boldsymbol{x_1} + \boldsymbol{x_2} + \boldsymbol{x_3} + \boldsymbol{1})$
- Subject to:
  - $\boldsymbol{x_1} G_k^n(\boldsymbol{d_1}) + \boldsymbol{x_2} G_k^n(\boldsymbol{d_2}) + \boldsymbol{x_3} G_k^n(\boldsymbol{d_3}) + \boldsymbol{1} \geq 0$

$\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$ 3 variables

Upper bound of spherical 3-distance set (LP)

- Object: $\boldsymbol{max(x_1 + x_2 + x_3 + 1)}$
- Subject to:
  - $\boldsymbol{x_1} G_k^n(\boldsymbol{F_1}(\boldsymbol{d_3})) + \boldsymbol{x_2} G_k^n(\boldsymbol{F_2}(\boldsymbol{d_3})) + \boldsymbol{x_3} G_k^n(\boldsymbol{d_3}) + \boldsymbol{1} \geq 0$

uni-variate $\mathbf{d_3}$

Upper bound of spherical 3-distance set (LP)

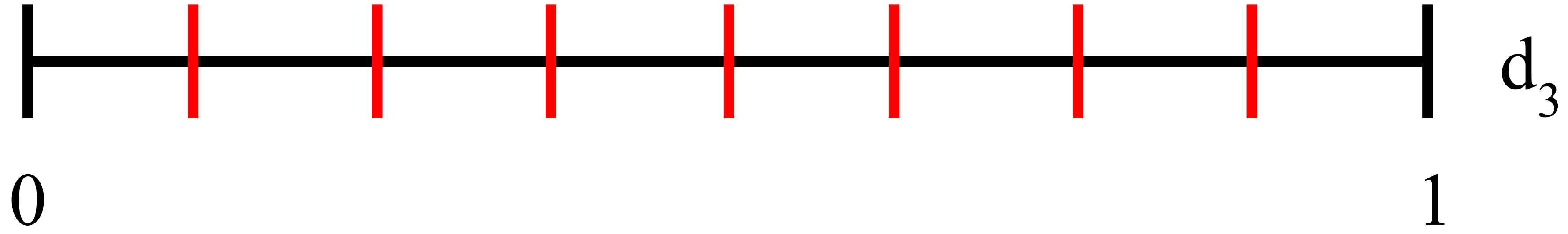
- Object: $\boldsymbol{max(x_1 + x_2 + x_3 + 1)}$
- Subject to:
  - $\boldsymbol{x_1} G_k^n(\boldsymbol{F_1(d_3)}) + \boldsymbol{x_2} G_k^n(\boldsymbol{F_2(d_3)}) + \boldsymbol{x_3} G_k^n(\boldsymbol{d_3}) + \boldsymbol{1} \geq 0$

uni-variate $\mathbf{d_3}$

Upper bound of spherical 3-distance set (LP)

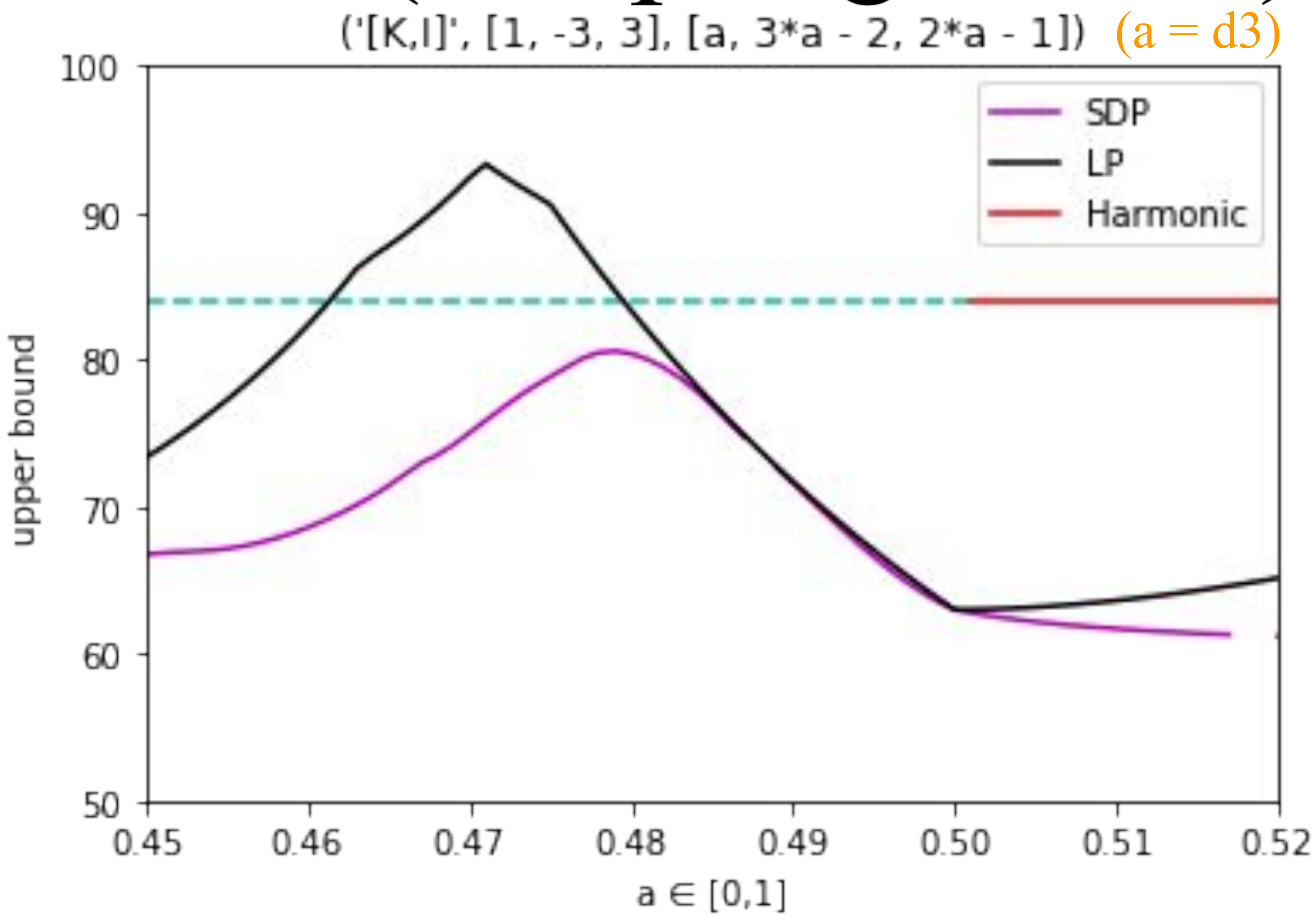
- Object: $\boldsymbol{max(x_1 + x_2 + x_3 + 1)}$
- Subject to:
  - $\boldsymbol{x_1} G_k^n(\boldsymbol{F_1(d_3)}) + \boldsymbol{x_2} G_k^n(\boldsymbol{F_2(d_3)}) + \boldsymbol{x_3} G_k^n(\boldsymbol{d_3}) + \boldsymbol{1} \geq 0$

uni-variate $\mathbf{d_3}$

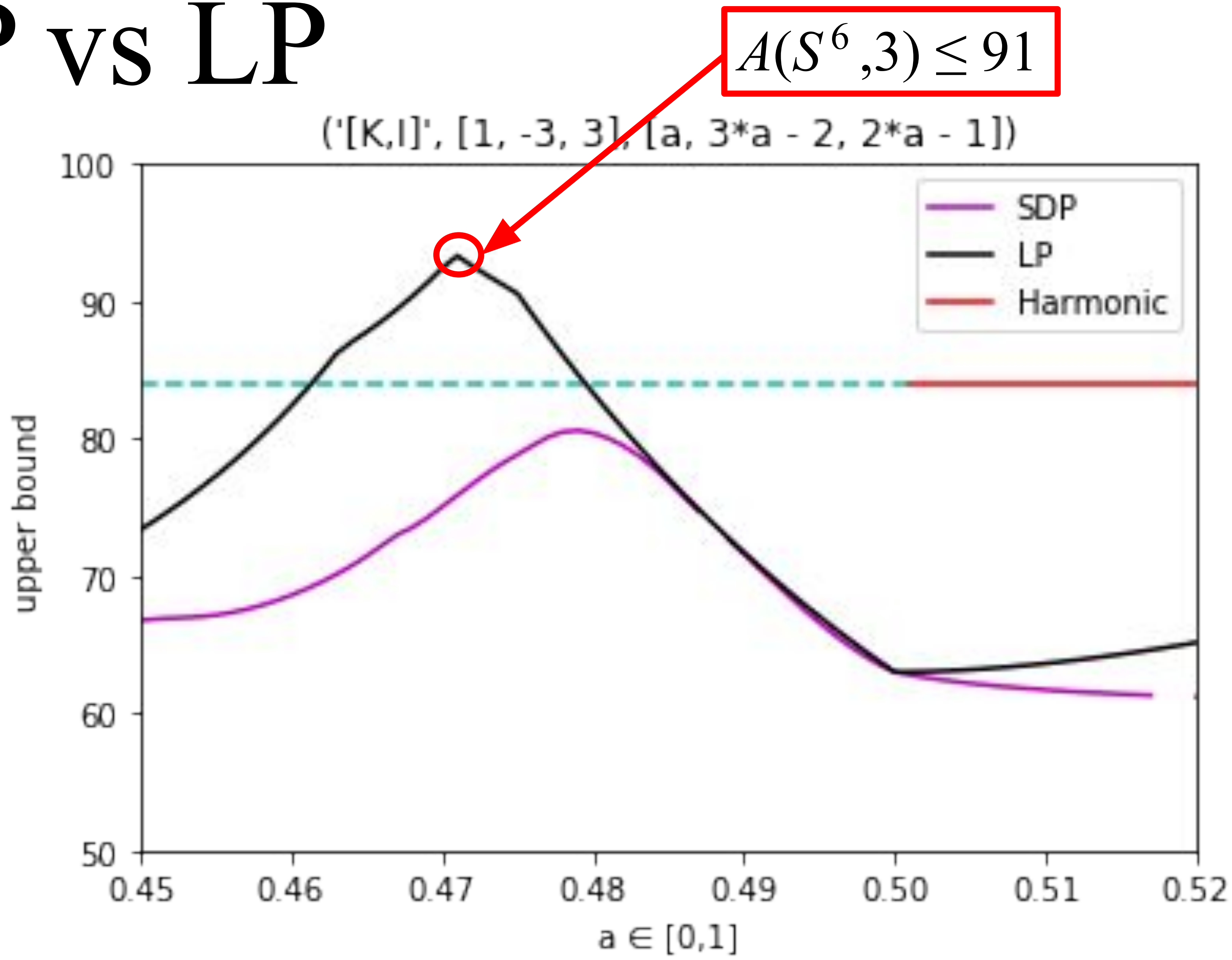


Lots of discrete sampling points

SDP vs LP (sampling on $\mathbb{R}^7$)

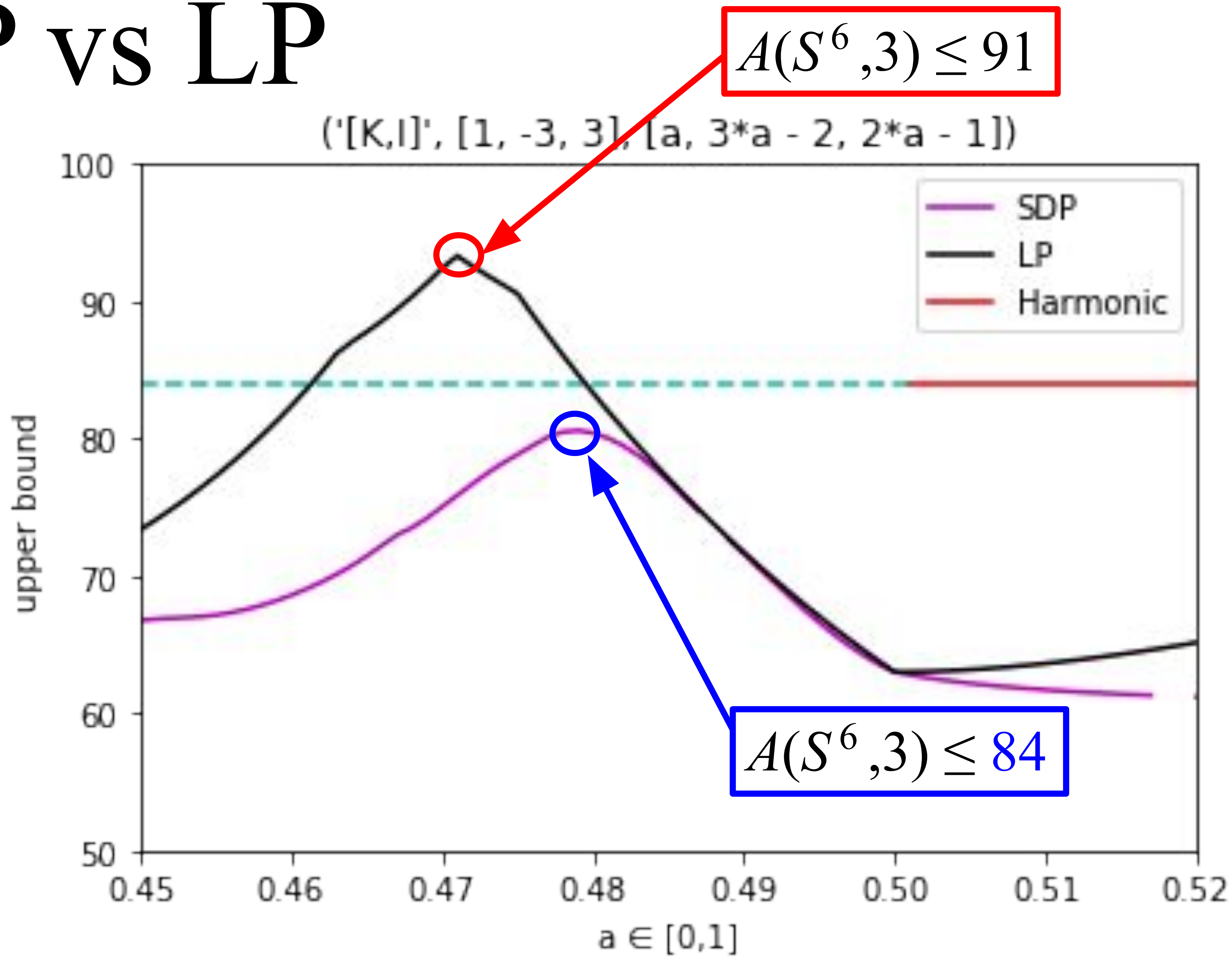


SDP vs LP



$A(S^6,3) \leq 91$
('[K,I]', [1, -3, 3], [a, 3*a - 2, 2*a - 1])
100
90
80
70
60
50
upper bound
0.45
0.46
0.47
0.48
0.49
0.50
0.51
0.52
a ∈ [0,1]
SDP
LP
Harmonic

SDP vs LP


$A(S^6,3) \leq 91$
('[K,I]', [1, -3, 3], [a, 3*a - 2, 2*a - 1])
100
90
80
70
60
50
upper bound
SDP
LP
Harmonic
0.45
0.46
0.47
0.48
0.49
0.50
0.51
0.52
a ∈ [0,1]
$A(S^6,3) \leq 84$

Outline

- Introduction & history

previous method

- Harmonic absolute bound
- Linear programming (LP)

our method

- Semidefinite programming (SDP)

experiment technique

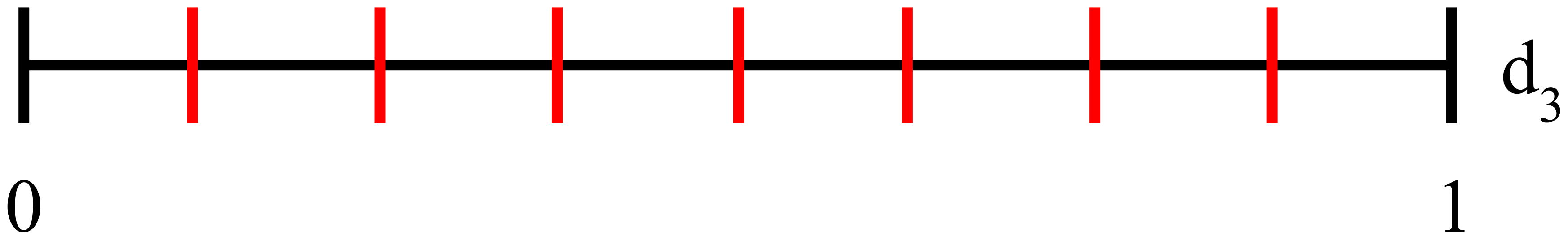
- Discrete sampling points with Nozaki theorem
- **Rigorous proof with sum of squares method (SOS)**

LP

SDP

Idea:

Divide the interval into many parts (p)

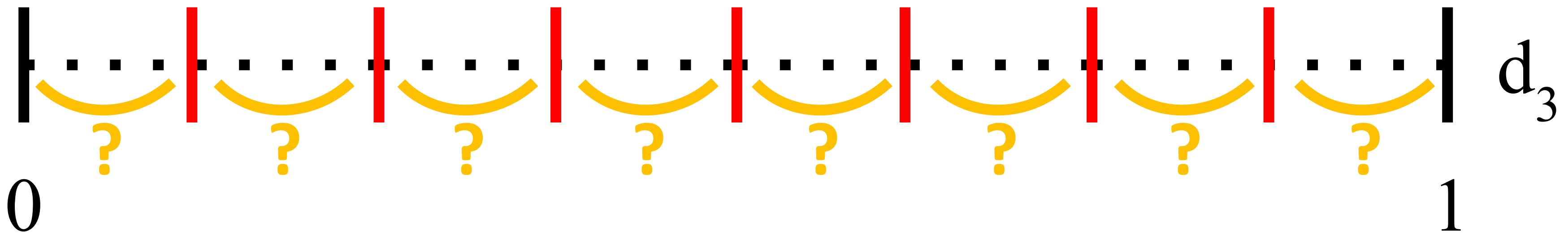


LP

SDP

Idea:

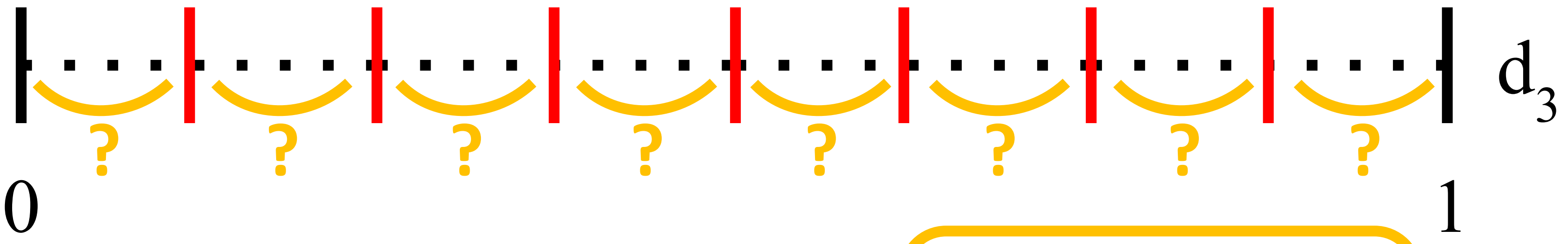
Divide the interval into many parts (p)



LP

SDP

Idea:
Divide the interval into many parts (p)



SOS

SDP primal form

$$\textit{maximize} \quad 1 + \frac{1}{3}(x_1 + x_2 + x_3)$$

$$\textit{subject to} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(x_1 + x_2 + x_3) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\sum_{i=4}^{13} x_i \succcurlyeq 0$$

$$3 + x_1 G_k^n(d_1) + x_2 G_k^n(d_2) + x_3 G_k^n(d_3) \geq 0,\ k = 1, 2, \cdots, p_{LP}$$

$$S_k^n(1,1,1) + x_1 S_k^n(d_1,d_1,1) + x_2 S_k^n(d_2,d_2,1) + x_3 S_k^n(d_3,d_3,1)$$

$$+ x_4 S_k^n(d_1,d_1,d_1) + x_5 S_k^n(d_2,d_2,d_2) + x_6 S_k^n(d_3,d_3,d_3)$$

$$+ x_7 S_k^n(d_1,d_1,d_2) + x_8 S_k^n(d_1,d_1,d_3) + x_9 S_k^n(d_2,d_2,d_1)$$

$$+ x_{10} S_k^n(d_2,d_2,d_3) + x_{11} S_k^n(d_3,d_3,d_1) + x_{12} S_k^n(d_3,d_3,d_2)$$

$$+ x_{13} S_k^n(d_1,d_2,d_3) \succcurlyeq 0,\ k = 0, 1, 2, \cdots, p_{SDP}$$

$$x_j \geq 0,\ j = 1, 2, \cdots, 13$$

SDP dual form

- Object: $$1 + min\{\sum_{i=1}^{p_{LP}} \alpha_i + \beta_{11} + \langle F_0, S_0^n(1,1,1)\rangle\}$$

- Subject to: $$\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{pmatrix} \succcurlyeq 0$$

$$\alpha_i \geq 0, i = 1, \cdots, p_{LP}$$
$$F_i \succcurlyeq 0, i = 0, \cdots, p_{SDP}$$

$$2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}} (\alpha_i G_i^n(d_1)) + \sum_{i=0}^{p_{SDP}} (3\langle F_i, S_i^n(d_1, d_1, 1)\rangle) \leq -1$$

$$2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}} (\alpha_i G_i^n(d_2)) + \sum_{i=0}^{p_{SDP}} (3\langle F_i, S_i^n(d_2, d_2, 1)\rangle) \leq -1$$

$$2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}} (\alpha_i G_i^n(d_3)) + \sum_{i=0}^{p_{SDP}} (3\langle F_i, S_i^n(d_3, d_3, 1)\rangle) \leq -1$$

$$\beta_{22} + \sum_{i=0}^{p_{SDP}} \langle F_i, S_i^n(y_1, y_2, y_3)\rangle \leq 0$$

$$(y_1, y_2, y_3) \in \{(d_1, d_1, d_1), (d_2, d_2, d_2), (d_3, d_3, d_3), (d_1, d_1, d_2), (d_1, d_1, d_3), (d_2, d_2, d_1), (d_2, d_2, d_3), (d_3, d_3, d_1), (d_3, d_3, d_2), (d_1, d_2, d_3)\}$$

SDP dual form

- Object: $$1 + min\{\sum_{i=1}^{p_{LP}} \alpha_i + \beta_{11} + \langle F_0, S_0^n(1,1,1)\rangle\}$$

- Subject to: $$\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{pmatrix} \succcurlyeq 0$$

$$\alpha_i \geq 0, i = 1, \cdots, p_{LP}$$
$$F_i \succcurlyeq 0, i = 0, \cdots, p_{SDP}$$

$$2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}} (\alpha_i G_i^n(d_1)) + \sum_{i=0}^{p_{SDP}} (3\langle F_i, S_i^n(d_1, d_1, 1)\rangle) \leq -1$$

$$2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}} (\alpha_i G_i^n(d_2)) + \sum_{i=0}^{p_{SDP}} (3\langle F_i, S_i^n(d_2, d_2, 1)\rangle) \leq -1$$

$$2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}} (\alpha_i G_i^n(d_3)) + \sum_{i=0}^{p_{SDP}} (3\langle F_i, S_i^n(d_3, d_3, 1)\rangle) \leq -1$$

$$\beta_{22} + \sum_{i=0}^{p_{SDP}} \langle F_i, S_i^n(y_1, y_2, y_3)\rangle \leq 0$$

$$(y_1, y_2, y_3) \in \{(d_1, d_1, d_1), (d_2, d_2, d_2), (d_3, d_3, d_3), (d_1, d_1, d_2), (d_1, d_1, d_3), (d_2, d_2, d_1), (d_2, d_2, d_3), (d_3, d_3, d_1), (d_3, d_3, d_2), (d_1, d_2, d_3)\}$$

SDP dual form

- Object: $1 + min\{\sum_{i=1}^{p_{LP}} \alpha_i + \beta_{11} + \langle F_0, S_0^n(1,1,1)\rangle\}$

- Subject to: $\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{pmatrix} \succcurlyeq 0$

$\alpha_i \geq 0, i = 1, \cdots, p_{LP}$

$F_i \succcurlyeq 0, i = 0, \cdots, p_{SDP}$

$$-\left(2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}}(\alpha_i G_i^n(d_1)) + \sum_{i=0}^{p_{SDP}}(3\langle F_i, S_i^n(d_1, d_1, 1)\rangle)\right) -1 \geq 0$$

$$-\left(2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}}(\alpha_i G_i^n(d_2)) + \sum_{i=0}^{p_{SDP}}(3\langle F_i, S_i^n(d_2, d_2, 1)\rangle)\right) -1 \geq 0$$

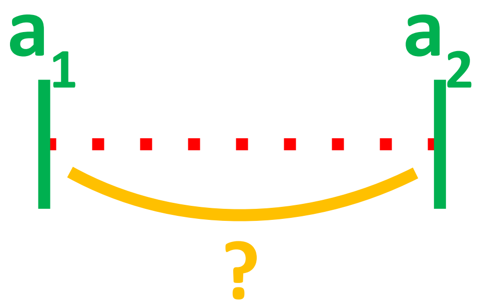
$$-\left(2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}}(\alpha_i G_i^n(d_3)) + \sum_{i=0}^{p_{SDP}}(3\langle F_i, S_i^n(d_3, d_3, 1)\rangle)\right) -1 \geq 0$$

$$-\left(\beta_{22} + \sum_{i=0}^{p_{SDP}}\langle F_i, S_i^n(y_1, y_2, y_3)\rangle\right) -1 \geq 0$$

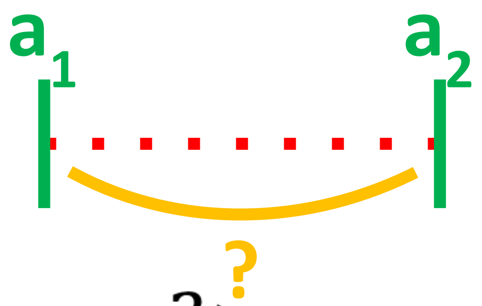
$(y_1, y_2, y_3) \in \{(d_1, d_1, d_1), (d_2, d_2, d_2), (d_3, d_3, d_3), (d_1, d_1, d_2), (d_1, d_1, d_3), (d_2, d_2, d_1), (d_2, d_2, d_3), (d_3, d_3, d_1), (d_3, d_3, d_2), (d_1, d_2, d_3)\}$

- $\forall a \in [a_1, a_2], \quad f(a) \geq 0$

  a := d3



- $\forall a \in [a_1, a_2],\ \ f(a) \geq 0$

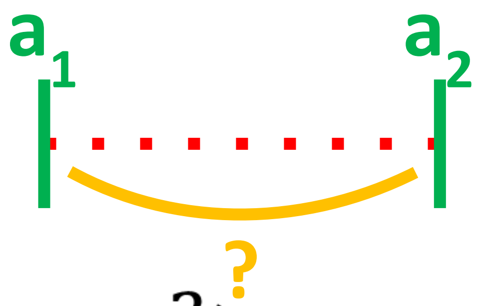


$$\Updownarrow$$

- $\forall x \in R,\ \ f^+(x) = (1 + x^2)^m f\left(\frac{a_1+a_2x^2}{1+x^2}\right) \geq 0$

$m = degree(f(a))$

- $\forall a \in [a_1, a_2],\ \ f(a) \geq 0$



$$\Updownarrow$$

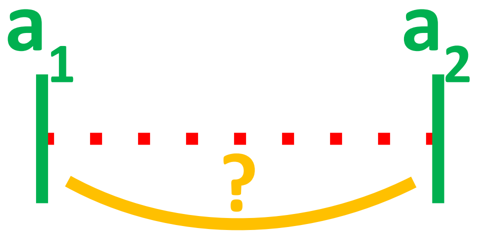
- $\forall x \in R,\ f^+(x) = (1 + x^2)^m f\left(\frac{a_1 + a_2 x^2}{1 + x^2}\right) \geq 0$

  $m = degree(f(a))$

$$\Updownarrow \text{ Hilbert}$$

- $f^+(x)$ can be written as Sum Of Square (SOS)

  $f^+(x) = \sum_i r_i^2(x)\,, r_i$ are polynomials

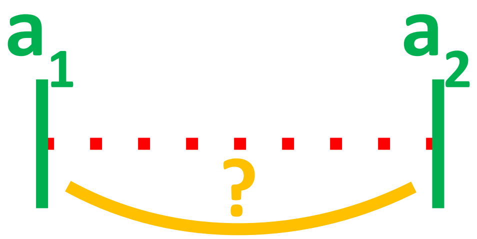


- $f^+(x)$ can be written as Sum Of Square (SOS)

$\Updownarrow$ Nesterov

- $\exists$ Q $(positive\ semidefinite\ matrix)$

s.t. $f^+ = XQX^t, X = (1, x, x^2, \dots, x^m)$



- $f^+(x)$ can be written as **Sum Of Square (SOS)**

$$\Updownarrow \text{ Nesterov}$$

- $\exists\, Q$ $(positive\ semidefinite\ matrix)$

  s.t. $f^+ = XQX^t, X = (1, x, x^2, \ldots, x^m)$

  $\Rightarrow$ **Semidefinite Matrix Condition!**

SDP dual form

- Object: $1 + min\{\sum_{i=1}^{p_{LP}} \alpha_i + \beta_{11} + \langle F_0, S_0^n(1,1,1)\rangle\}$
- Subject to: $\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{pmatrix} \succcurlyeq 0$

$\alpha_i \geq 0, i = 1, \cdots, p_{LP}$

$F_i \succcurlyeq 0, i = 0, \cdots, p_{SDP}$

f(a)

$$-\left(2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}}(\alpha_i G_i^n(d_1)) + \sum_{i=0}^{p_{SDP}}(3\langle F_i, S_i^n(d_1, d_1, 1)\rangle)\right) - 1 \geq 0$$

$$-\left(2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}}(\alpha_i G_i^n(d_2)) + \sum_{i=0}^{p_{SDP}}(3\langle F_i, S_i^n(d_2, d_2, 1)\rangle)\right) - 1 \geq 0$$

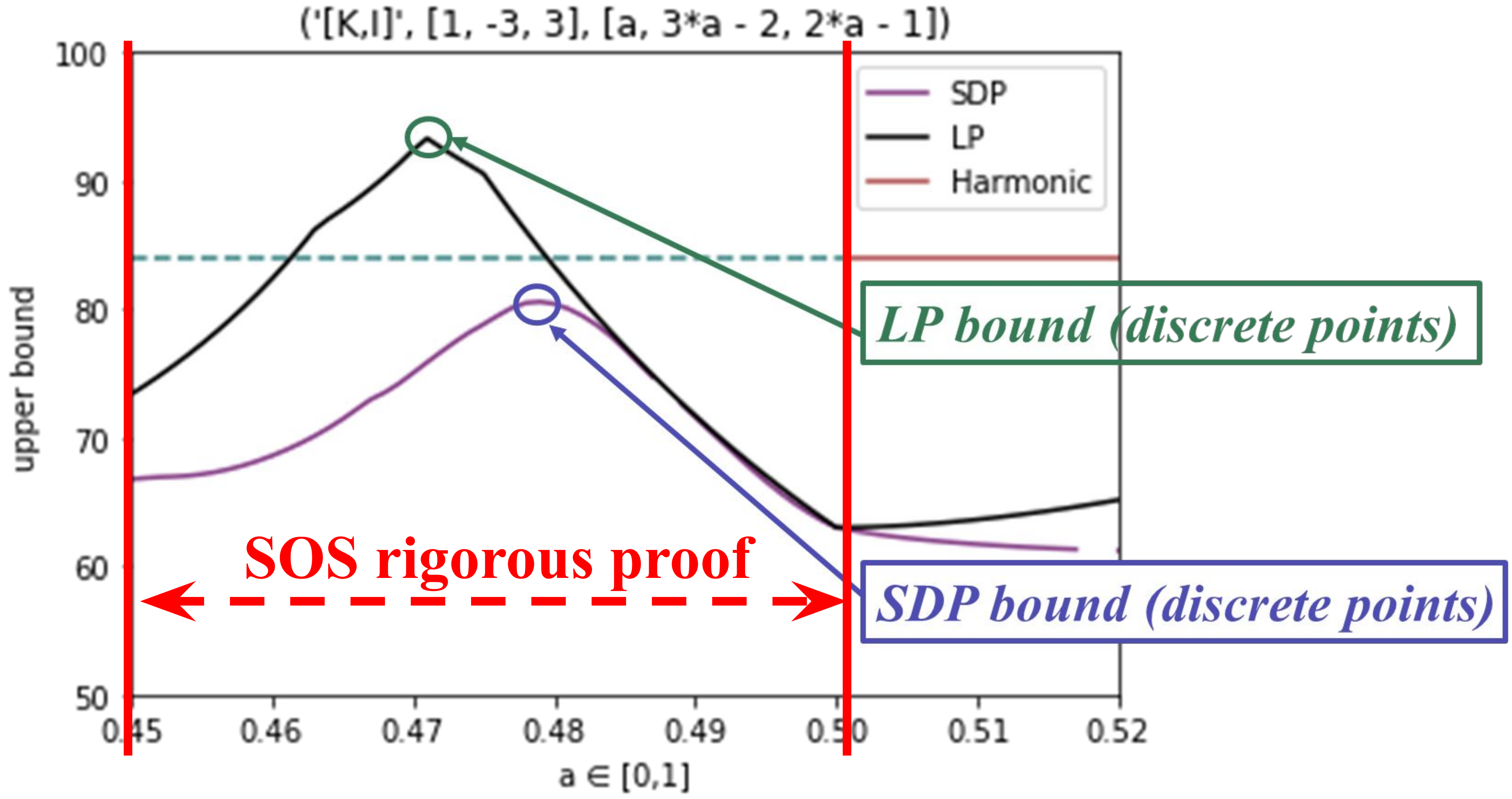
$$-\left(2\beta_{12} + \beta_{22} + \sum_{i=1}^{p_{LP}}(\alpha_i G_i^n(d_3)) + \sum_{i=0}^{p_{SDP}}(3\langle F_i, S_i^n(d_3, d_3, 1)\rangle)\right) - 1 \geq 0$$

$$-\left(\beta_{22} + \sum_{i=0}^{p_{SDP}}\langle F_i, S_i^n(y_1, y_2, y_3)\rangle\right) - 1 \geq 0$$

Semidefinite Matrix Condition!

$(y_1, y_2, y_3) \in \{(d_1,d_1,d_1), (d_2,d_2,d_2), (d_3,d_3,d_3), (d_1,d_1,d_2), (d_1,d_1,d_3), (d_2,d_2,d_1), (d_2,d_2,d_3), (d_3,d_3,d_1), (d_3,d_3,d_2), (d_1,d_2,d_3)\}$

Discrete vs Continuous



SOS Experiment

range of a	sos bound	range of a	sos bound	range of a	sos bound
[0.45, 0.451]	67.61	[0.467, 0.468]	74.59	[0.484, 0.485]	78.33
[0.451, 0.452]	67.75	[0.468, 0.469]	75.24	[0.485, 0.486]	77.18
[0.452, 0.453]	67.89	[0.469, 0.47]	76.01	[0.486, 0.487]	76.07
[0.453, 0.454]	68.01	[0.47, 0.471]	76.85	[0.487, 0.488]	75
[0.454, 0.455]	68.14	[0.471, 0.472]	77.71	[0.488, 0.489]	73.96
[0.455, 0.456]	68.3	[0.472, 0.473]	78.58	[0.489, 0.49]	72.94
[0.456, 0.457]	68.51	[0.473, 0.474]	79.45	[0.49, 0.491]	71.96
[0.457, 0.458]	68.79	[0.474, 0.475]	80.25	[0.491, 0.492]	71
[0.458, 0.459]	69.14	[0.475, 0.476]	80.99	[0.492, 0.493]	70.08
[0.459, 0.46]	69.55	[0.476, 0.477]	81.59	[0.493, 0.494]	69.17
[0.46, 0.461]	70.02	[0.477, 0.478]	81.96	[0.494, 0.495]	68.3
[0.461, 0.462]	70.55	[0.478, 0.479]	82.1	[0.495, 0.496]	67.42
[0.462, 0.463]	71.15	[0.479, 0.48]	82.06	[0.496, 0.497]	66.56
[0.463, 0.464]	71.81	[0.48, 0.481]	81.82	[0.497, 0.498]	65.72
[0.464, 0.465]	72.55	[0.481, 0.482]	81.37	[0.498, 0.499]	64.9
[0.465, 0.466]	73.37	[0.482, 0.483]	80.59	[0.499, 0.5]	64.1
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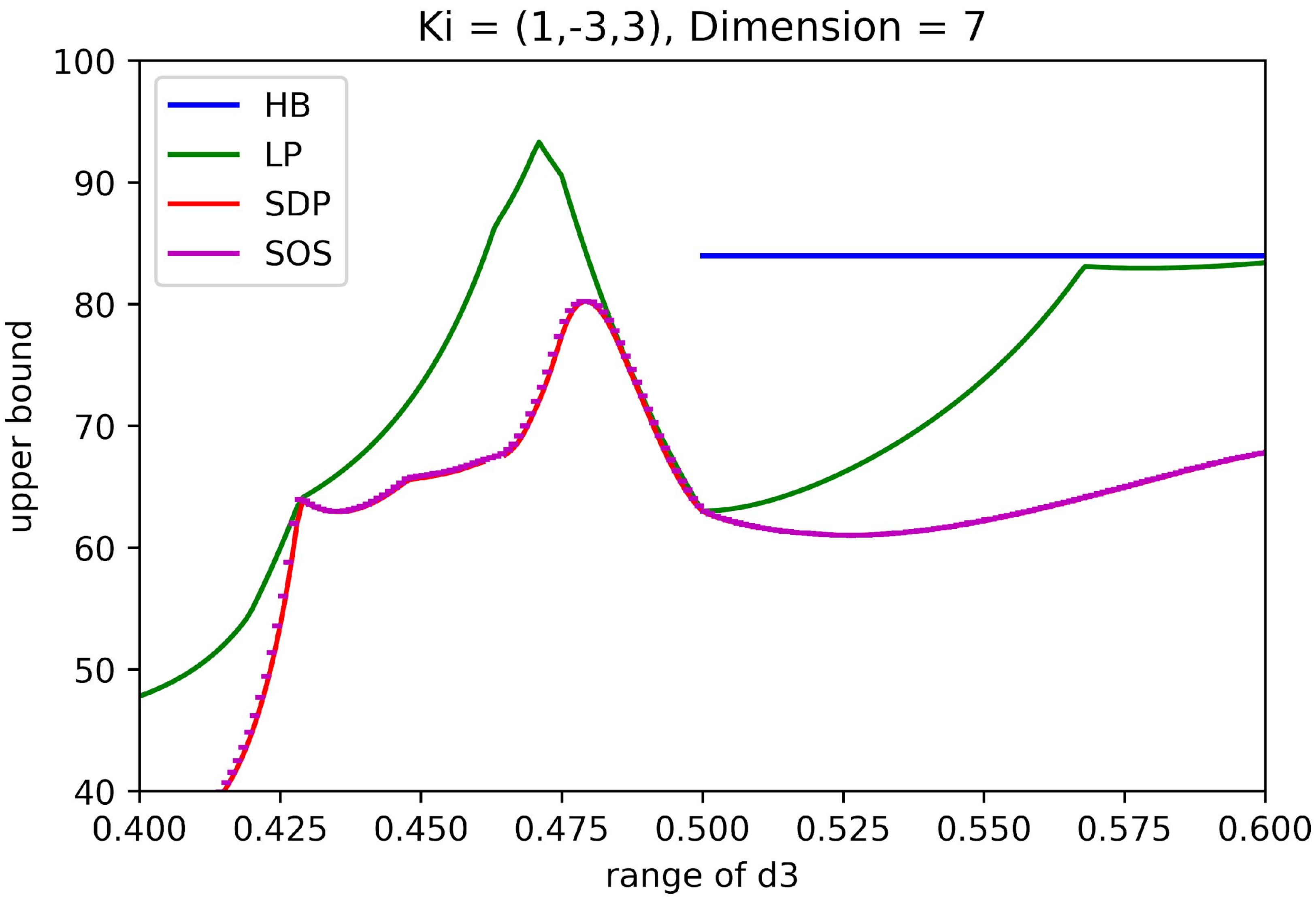
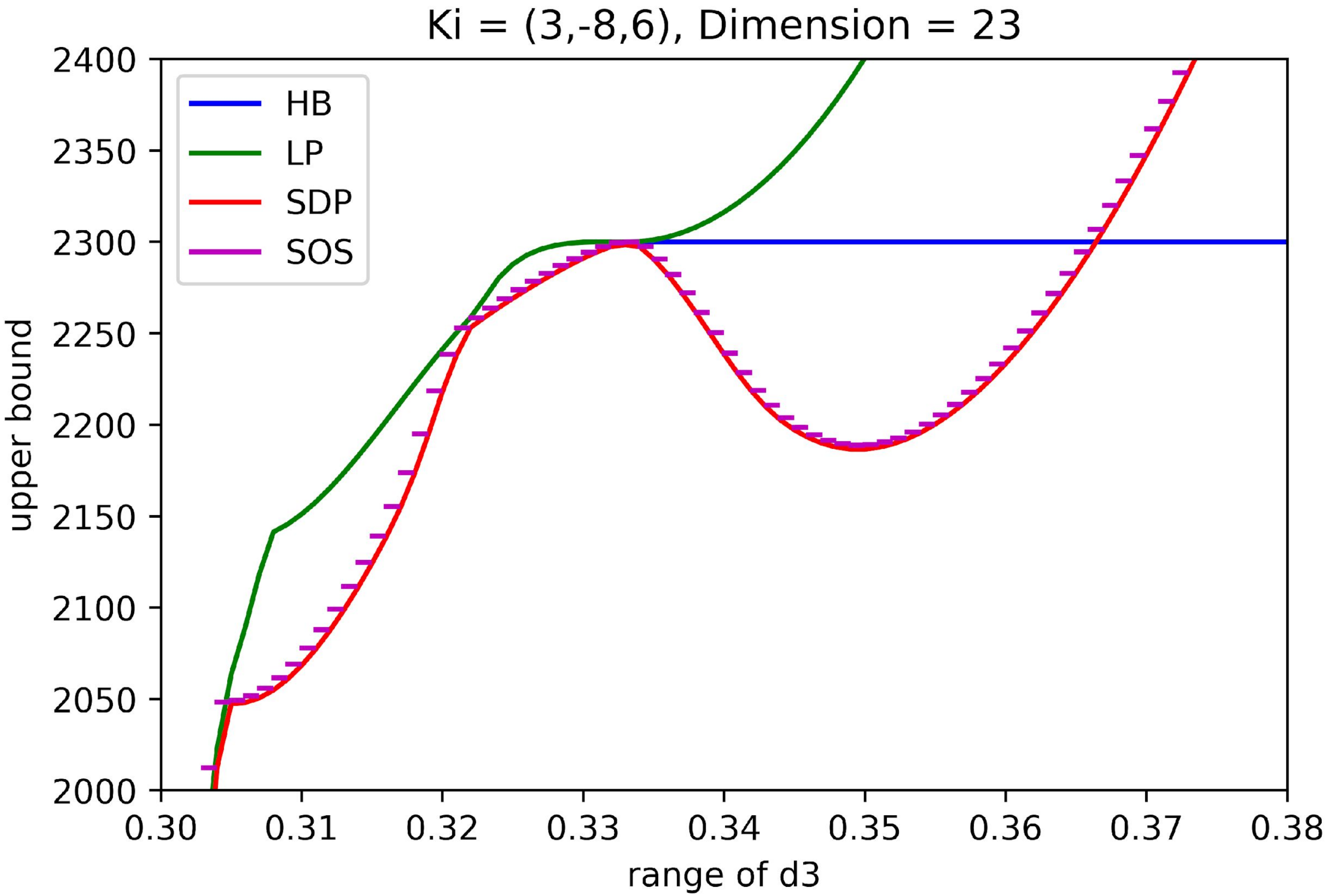
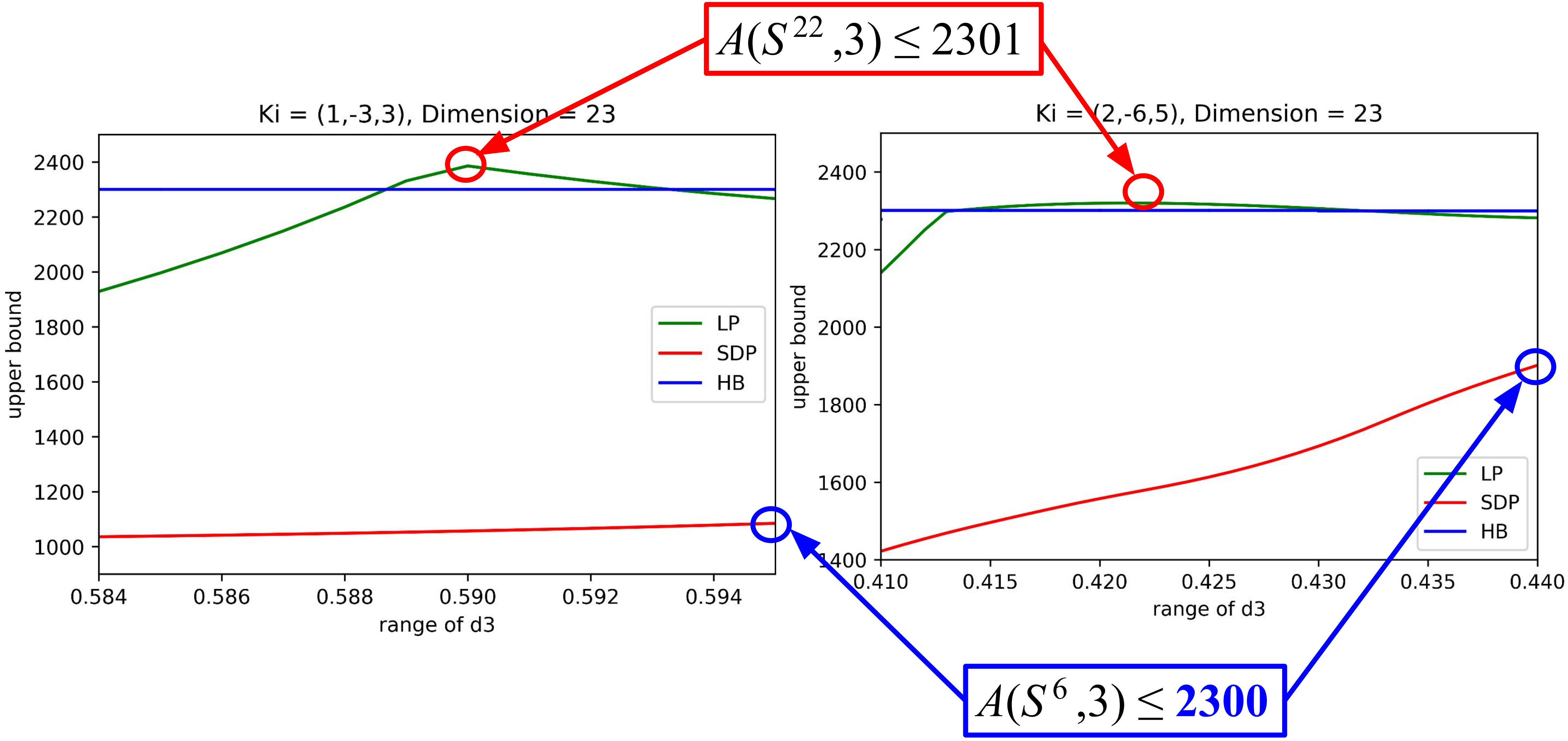
82.1 < harmonic bound = 84

$\mathbb{R}^7$: $A(S^6, 3) \leq 84$



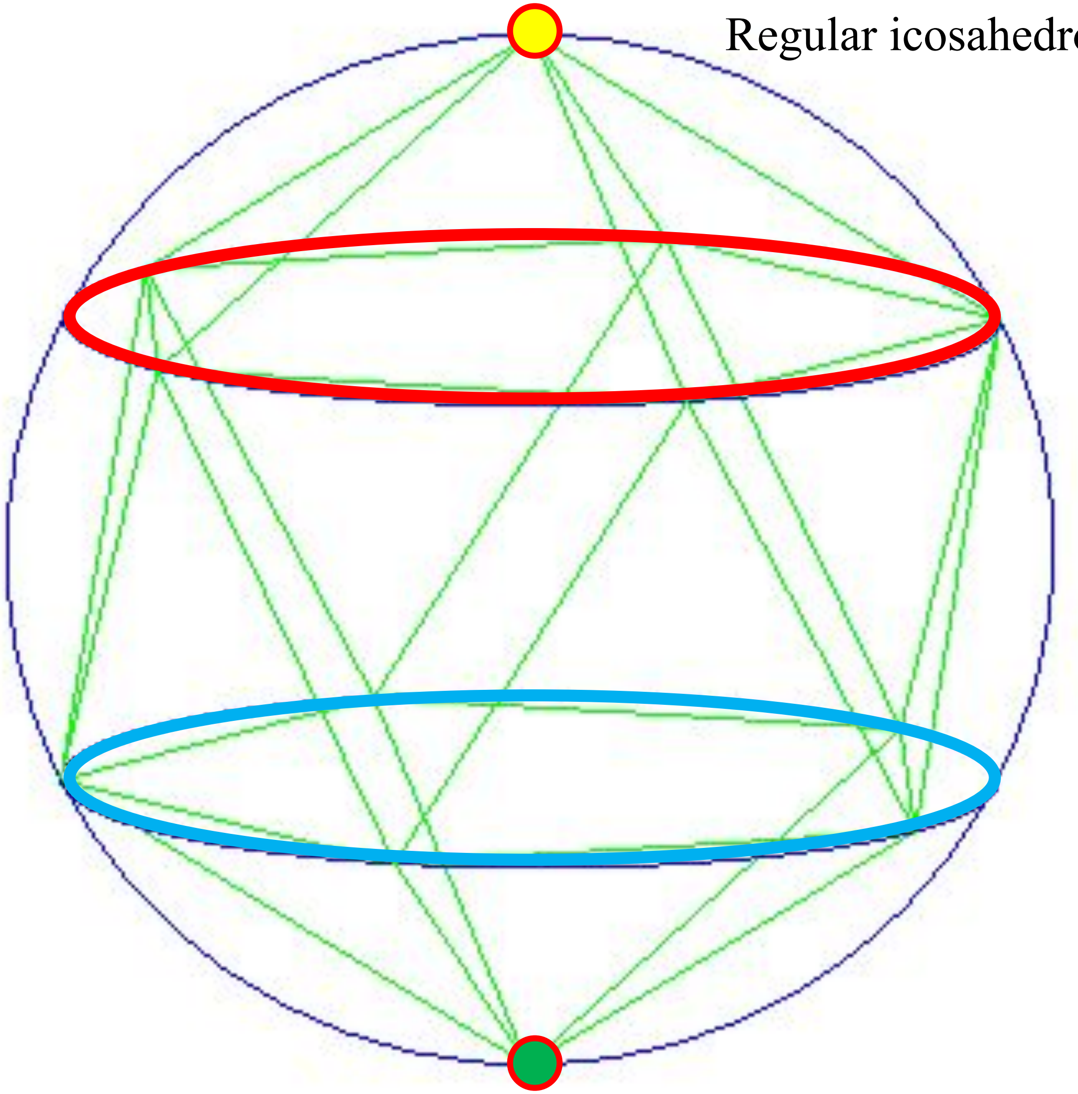
$\mathbb{R}^{23}$: $A(S^{22}, 3) = 2300$




$A(S^{22},3) \leq 2301$
Ki = (1,-3,3), Dimension = 23
Ki = (2,-6,5), Dimension = 23
upper bound
range of d3
LP
SDP
HB
2400
2200
2000
1800
1600
1400
1200
1000
0.584
0.586
0.588
0.590
0.592
0.594
0.410
0.415
0.420
0.425
0.430
0.435
0.440
$A(S^{6},3) \leq$ **2300**

Summary

- Spherical 3-distance set



Summary

- Spherical 3-distance set

- LP, SDP

$$\sum_{(x,y)\in C^2} G_k^n(x\cdot y) \geq 0$$

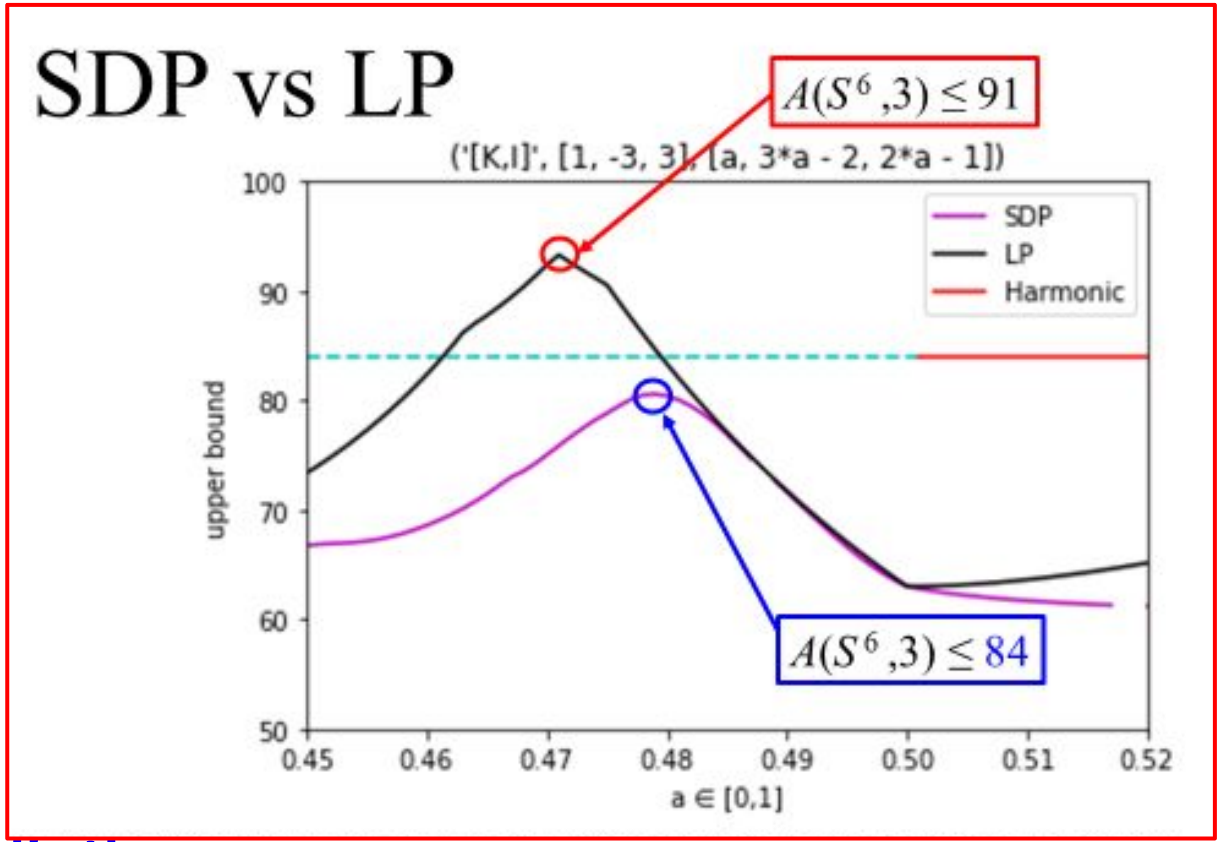
Delsarte
(LP)

$$\sum_{(x,y,z)\in C^3} S_k^n(x\cdot y, x\cdot z, y\cdot z) \succcurlyeq 0$$

Schoenberg
(SDP)

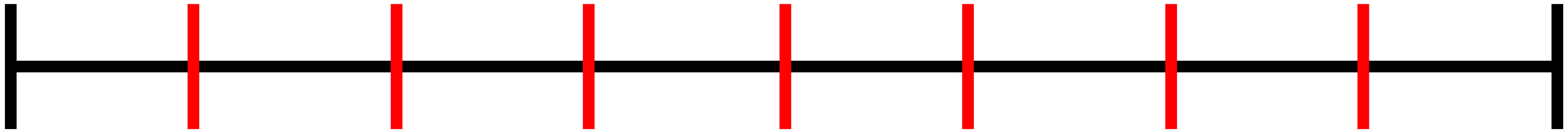
Summary

- Spherical 3-distance set
- LP, SDP
- Nozaki theorem



Generalization of Larman-Rogers-Seidel's theorem

$(d_1,d_2,d_3) \to (F_1(d_3),F_2(d_3),d_3)$

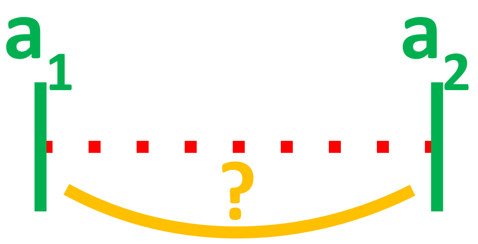


d_3 (a)

0 (Lots of discrete sampling points) 1

Summary

- Spherical 3-distance set
- LP, SDP
- Nozaki theorem
- Sum of squares



Summary

- Spherical 3-distance set

- LP, SDP

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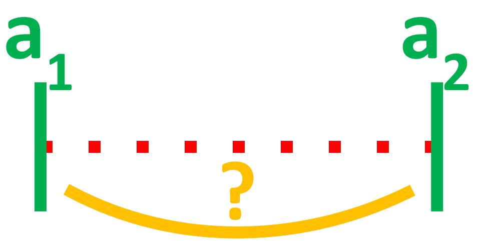

Max Spherical *3*-distance set in R^7: upper bound 91→**84** (discrete &**rigorous proof**)

Max Spherical *3*-distance set in R^{23}: **2300**, a half of tight spherical 7-design

Thanks for your listening ☺