Rainbow Graph Designs

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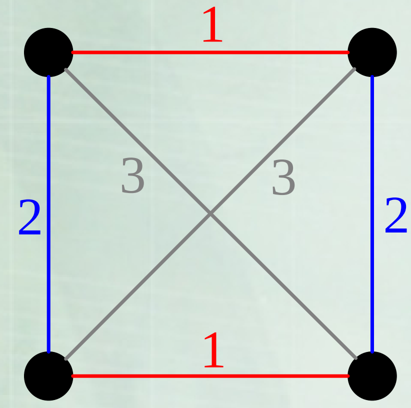
Taipei City with two rainbows

Preliminaries

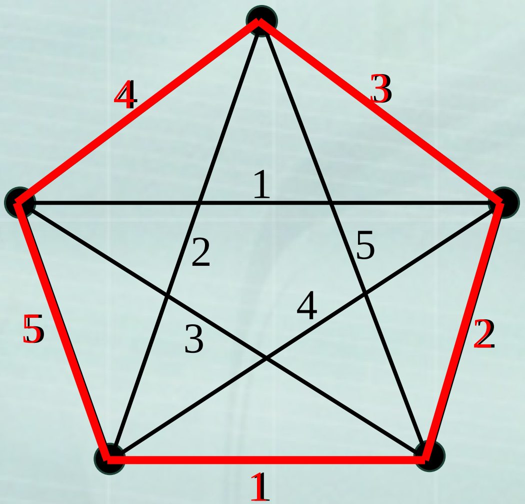
- A (proper) ***k-edge-coloring*** of a graph G is a mapping from $E(G)$ into $\{1,\ldots,k\}$ such that incident edges of G receive (distinct) colors.
- The ***chromatic index*** of G, denoted by $\chi'(G)$, is the minimum number k such that G has a k-edge-coloring.
- In this talk, all colorings we mention are ***proper***.

A proper 3-edge-coloring of K_4



Rainbow subgraph

- A subgraph H in an edge-colored graph G is a ***rainbow subgraph*** of G if no two edges in H have the same color.



A rainbow 5-cycle

Monochromatic Subgraphs

- A subgraph of an edge-colored graph is a *monochromatic subgraph* if all its edges are of the same color.
- Clearly, if the edge-coloring is proper, then all monochromatic subgraphs are matchings.
- But, if the edge-coloring is not a proper coloring, then "*many things*" could happen!

Graph Decomposition

- An *H-packing* of a graph G is a collection of edge-disjoint subgraphs of G such that each of them is isomorphic to H.
- G is an *H-decomposition* of G if G has an H-packing such that all edges of G have been used in the packing.
- We use H│G to denote the decomposition.
- If G is the complete graph of order n, then H│G is known as an *H-design of order n*.

Rainbow Graph Designs

- We are looking for as many isomorphic rainbow subgraphs inside an edge-colored graph as possible.
- So, given a graph G with an edge-coloring either prescribed or arbitrarily given, we try to pack the graph G with rainbow subgraphs which are isomorphic to H.
- If we can use up all the edges of the graph G such that each edge of G occurs in exactly one H, then we have a rainbow H-design of G.

Weak and Strong Designs

- If we can find a $\chi'(G)$-edge-coloring of G and then decompose G into rainbow subgraphs H, then we have a *weak rainbow H design of G*, denoted by $H|_r G$.
- On the other hand, if for any proper edge-coloring of G, we can decompose the G into rainbow subgraphs H, then we have a *strong rainbow H design of G*, denoted by $H|_R G$.
- If G is the complete graph of order n, then the designs will be referred to as rainbow design of order n respectively (weak or strong).

Motivation: Rainbow Is Beautiful!

More works are on mono-chromatic subgraphs!

Rainbow 1-factor

- **Theorem** (Woolbright and Fu, 1998)

  In any (2m-1)-edge-colored K_{2m} where $m > 2$, there exists a rainbow 1-factor.

- **Open problem**

  Find *two* or more edge-disjoint rainbow 1-factors in any (2m-1)-edge-colored K_{2m} where $m > 2$?

- Note that there exists a (2m – 1)-edge-coloring of K_{2m} such that there are 2m – 1 rainbow 1-factors in K_{2m} for $m \geq 8$. (?)

Room Squares

- A Room square of side 2m-1 provides a (2m-1)-edge-coloring of K_{2m} such that 2m-1 edge-disjoint multicolored 1-factors exist.

			35	**17**	**28**	**46**
	26	48			15	37
	13	57	68	24		
47		16		38		25
58		23	14		67	
12	78			56	34	
36	45		27			18

m = 4

Rainbow 1-factor Design of order 2m

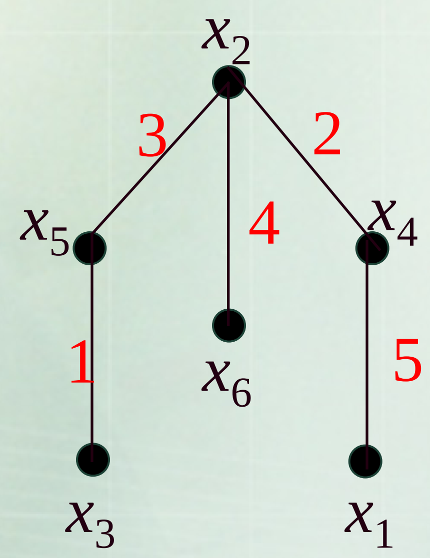
- Such *weak rainbow 1-factor design of order 2m* exists following the construction of a Room squares of order 2m.
- It was proved (combined all the early works) by W. Wallis that except for m = 2, 3, a Room square of order 2m exists.
- But, it is going to be very hard to show that a strong rainbow 1-factor design of order 2m does exist.

How about graphs with more edges?

- Since the edge-colorings we consider are proper, the cases of triangle and stars are trivial as long as we have a design of feasible orders.
- From the conjecture mentioned in next slide, we first consider the spanning tree case.

Isomorphic Rainbow trees in K_6

	T_1	T_2	T_3
Color 1	x_3x_5	x_4x_6	x_1x_2
Color 2	x_2x_4	x_1x_5	x_3x_6
Color 3	x_2x_5	x_3x_4	x_1x_6
Color 4	x_2x_6	x_1x_3	x_4x_5
Color 5	x_1x_4	x_2x_3	x_5x_6



T_1

Spanning Trees

- **Constantine's Weak Conjecture** (2002)
  For any $m > 2$, K_{2m} can be $(2m-1)$-edge-colored in such a way that the edges can be partitioned into m isomorphic rainbow spanning trees.
- This conjecture was verified later. So, we do have a weak rainbow spanning-tree design of order 2m for each m ≥ 3 for "certain spanning tree", see next slide for an example of m = 5.
- How about the other spanning trees?

(*) S. Akbari, A. Alipour, H. L. Fu and Y. H. Lo, Multicolored parallelisms of isomorphic spanning trees, SIAM J. Discrete Math. 20 (2006), No. 3, 564-567.

An example: m = 5

Impossible Mission

- **Constantine's Strong Conjecture** (2002)
  If $m > 2$, then in any proper $(2m-1)$-edge-coloring of K_{2m}, all edges can be partitioned into m isomorphic rainbow spanning trees.
- Brualdi-Hollingsworth's Conjecture (1996)
  If $m > 2$, then in any proper $(2m-1)$-edge-coloring of K_{2m}, all edges can be partitioned into m rainbow spanning trees.

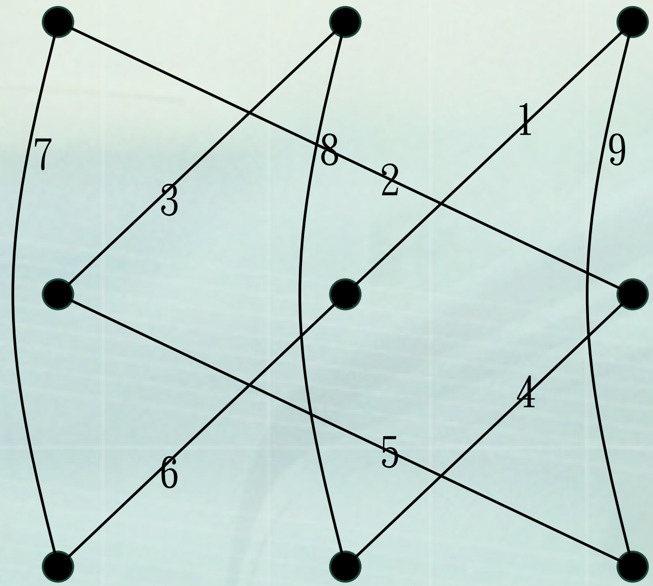
(*) R. A. Brualdi and S. Hollingsworth, Multicolored trees in complete graphs, J. Combin. Theory Ser. B 68 (1996), No. 2, 310-313.

Weak Rainbow Hamilton-cycle Design

- Conjecture: there exists a proper $(2m+1)$-edge-coloring of K_{2m+1} for which all edges can be partitioned into m isomorphic rainbow spanning unicyclic subgraphs.
- Yes, there exists one such edge-coloring and proper subgraphs (Hamiltonian cycles). See an example of m = 4 in next slide.

(*) H. L. Fu and Y. H. Lo, Multicolored parallelisms of Hamiltonian cycles, Discrete Math. 309 (2009), No. 14, 4871-4876.

Rainbow Hamilton cycles



Bipartite difference = 2

Bipartite difference = 2

Bipartite difference = 0

Strong Rainbow Designs

Constantine's Strong Conjecture on odd order

(2005)

In any proper $(2m+1)$-edge-coloring of K_{2m+1}, all edges can be partitioned into m rainbow isomorphic spanning unicyclic subgraphs.

- If arbitrary coloring is considered, then finding one rainbow Hamilton cycle (or Hamilton path) is already a very difficult job!

Rainbow Cycle Designs

- We expect the following result:

  If $C_k \mid K_n$, then there exists an n-edge-coloring of K_n such that $C_k \mid_r K_n$.
- The cases k = 3 and k = n (odd) are true.
- The following idea shows that for $k = 2^t$, we also have $C_k \mid_r K_n$.

Edge-colorings to Use

- Let $V(K_{2m+1}) = \{v_i \mid i \in Z_{2m+1}\}$.
- Let $\varphi(v_i v_j) \equiv i + j \pmod{2m+1}$. Then φ is a proper (2m+1)-edge-coloring of K_{2m+1}. Note that the chromatic index of K_{2m+1} is 2m+1.
- Let $V(K_{m,m}) = A \cup B$ where $A = \{a_i \mid i \in Z_m\}$ and $B = \{b_i \mid i \in Z_m\}$.
- Let $\pi(a_i b_j) \equiv j - i \pmod{m}$. Then π is a proper m-edge-coloring of $K_{m,m}$.

Graph Decomposition

- We shall decompose the graph cyclically by using the so-called *difference method*.
- This idea was introduced by A. Rosa in early 60's.
- The difference of v_i and v_j in $V(K_{2m+1})$ is defined as $\min\{|j - i|, (2m+1) - |j - i|\}$. (It is also known as the half difference.)

A. Rosa, on cyclic decompositions of complete graph into polygons with an odd number of edges, Casopis Pest. Math. 91 (1966), 53 – 63.

Labeling

- A vertex labeling Ψ of G is an assignment of V(G) by using the labels in {0, 1, 2, …, k} such that each label occurs at most once. For convenience, it is called a k-vertex labeling.
- The weight of an edge in a graph with k-vertex labeling is defined as the (positive) difference of its two end vertices.
- A k-vertex labeling of a graph G is a *graceful* (β) labeling if k is the size of the graph and all the weights obtained are distinct.

Labeling-Continued

- If k is 2|E(G)| + 1 and the weights obtained are 1, 2, …, |E(G)|, then we have a *ρ-labeling*.
- A *ρ-labeling* is a *bipartite ρ-labeling* if there exists a λ such that for each edge exactly one of the two vertices receives a labels at most λ.
- For example, let G be a 4-cycle and (0,4,2,3) is a 4-vertex labeling of G. Then this labeling is a graceful labeling. If G is a subgraph of K_9, then we may label G with k = 9, say (2,6,4,5). Again, all weights are distinct. So, it is a *ρ-labeling, in fact it is a bipartite ρ-labeling by letting* λ = 4.

Beautiful Decomposition

- Theorem (A. Rosa)

  Let H be a graph of size k and H has a *ρ-labeling* using colors 0, 1, 2, …, 2k. Then $H \mid K_{2k+1}$. Moreover, if *ρ* is a bipartite labeling, then $H \mid K_{2tk+1}$.

Example

$C_4 \mid K_9$ and also $C_4 \mid K_{8t+1}$. (0,4,2,3) is a bipartite *ρ-labeling.*

A. Rosa, on cyclic decompositions of complete graph into polygons with an odd number of edges, Casopis Pest. Math. 91 (1966), 53 – 63.

Weak Rainbow C_4 Designs

- Consider K_9 defined on $\{v_i \mid i \in Z_9\}$ is edge-colored by using the edge-coloring mentioned earlier: $\varphi(v_i v_j) \equiv i + j \pmod{2m+1}$, m = 4.
- The edges of (v_0, v_4, v_2, v_3) are of colors 4, 6, 5 and 3 respectively.
- Now, we can decompose K_9 by difference method, shift (v_0, v_4, v_2, v_3) to obtain a weak rainbow 4-cycle design. (v_i to $v_{i+1 \pmod 9}$)
- Subsequently, we also have $C_4 \mid_r K_{8t+1}$ for each positive integer t.

Another Example

- $K_4 \mid_r K_{13}$.
- Consider K_{13} defined on $\{v_i \mid i \in Z_{13}\}$ is edge-colored by using the edge-coloring mentioned earlier: $\varphi(v_i v_j) \equiv i + j \pmod{2m+1}$, $m = 6$.
- Use the K_4 induced by $\{v_0, v_1, v_3, v_9\}$ to generate the decomposition.
- Differences are 1, 2, …, 6 and colors on the graph are 1, 3, 4, 9, 10, 12.
- After one shift, colors are changing but the graph remains a rainbow.

General Idea

- If we would like to apply difference method to find a weak rainbow H-design of order $2 \cdot |E(H)| + 1$, then we need to find a special ρ-labeling such that the sums (of two ends of edges) are also distinct modulo

  $2 \cdot |E(H)| + 1$.
- Of course, the edge-coloring used here is not the only one, so is the labeling. There are many other choice!
- For example, the one used in obtaining weak rainbow Hamilton cycle design is different.

Weak Rainbow Cycle Designs

- $C_4 \mid {}_rK_{8t+1}$ shows that for each admissible order of 4-cycle design of order 8t + 1 we can obtain a weak rainbow 4-cycle design.
- How about the other cycle designs of admissible orders?
- Some works have been done for cycle length k ≡ 0 or 3.

Rainbow cycle designs, Bulletin of The ICA, Volume 81 (2017), 118 – 130.

Strong Rainbow 4-cycle Designs

- Now, the edge coloring is arbitrarily given, but it is a proper coloring.
- Theorem

  For each $m \geq 3$, there exists a strong rainbow 4-cycle design of $K_{2m,2m}$.

  Proof. First, we show that $C_4 |_R K_{2,2m}$ for $m \geq 3$. Then, the proof follows.
- So, by n to n + 8 construction of 4-cycle systems and $C_4 |_R K_9$ we can conclude that $C_4 |_R K_{8t+1}$.

The Missing Piece

- Problem: Show that $C_4 | _R K_9$.
- Theorem For each $t \geq 3$, $C_4 | _R K_{8t+1}$.

  Proof. By the fact that $K_{2, 2t}$ has a bipartite ρ-labeling defined on Z_{8t+1} we can decompose K_{8t+1} into copies of $K_{2, 2t}$ and conclude the proof.
- Since $C_4 | _R K_{2,2t}$ for $t \geq 3$, we can not conclude the proof for $t = 1, 2$. (Too bad!)

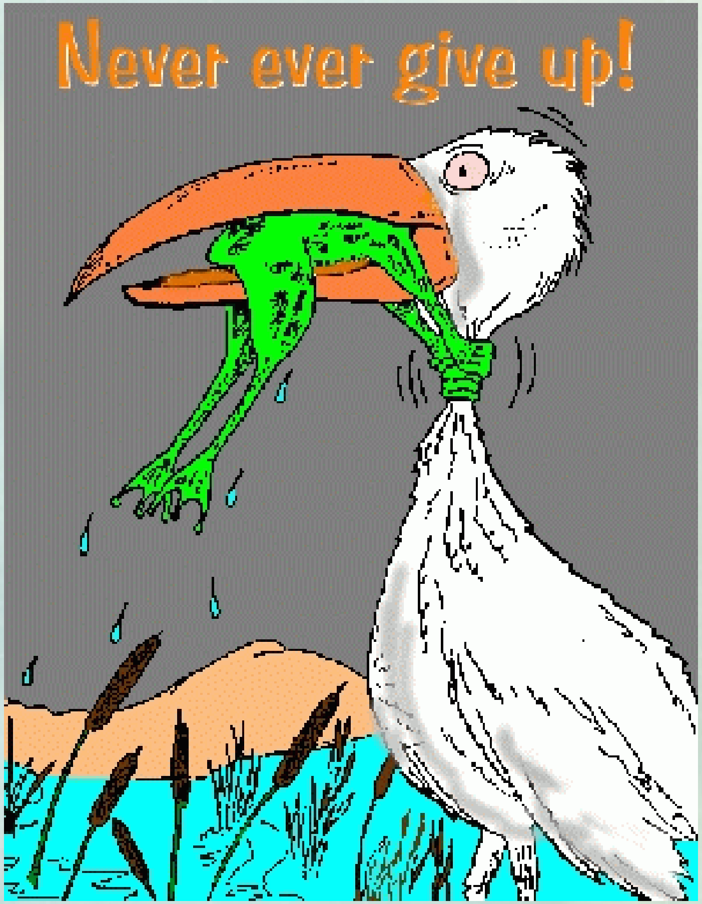
Further Try

- We may use a similar argument to show that $P_4 |_R K_n$ provided $n \equiv 0$ or $1 \pmod 3$ and $n \geq 6$.
- Moreover, if $n \equiv 2$, then we are able to pack K_n with maximum number of rainbow P_4's.
- We believe that more works can be done.

- Strictly rainbow 4-cycle designs (work jointly with Jun-yi Kuo and Zhen-Jun Chen), in preprints.

Don’t Stop!

We Have to Stop!



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