Folding Phenomenon of Major-balance Identities on Restricted Involutions

Tung-Shan Fu

National Pingtung University

Tenth Cross-strait Conference on Graph Theory and Combinatorics
National Chung-Hsing University, Taichung, August 18-23, 2019

Acknowledgement of collaboration

- Sen-Peng Eu, T.-S. Fu, Yeh-Jong, and Chien-Tai Ting, Sign-balance identities of Adin-Roichman type on 321-avoiding alternating permutations, Discrete Math. 312 (2012), 2228–2237.

- Sen-Peng Eu, T.-S. Fu, Yeh-Jong Pan, A refined sign-balance of simsun permutations, European J. Comb. 36 (2014), 97–109.

- Sen-Peng Eu, T.-S. Fu, Yeh-Jong Pan, Chien-Tai Ting, Two refined major-balance identities on 321-avoiding involutions, European J. Comb. 49 (2015), 250–264.

- T.-S. Fu, Hsiang-Chun Hsu, Hsin-Chieh Liao, Folding Phenomenon of Major-balance Identities on Restricted Involutions, Taiwanese J. Math. 22 (2018) 1031–1050.

outline

- introduction of the notion of sign-balances
- previously known results with folding phenomenon
- refined major-balance results on 321-avoiding (123-avoiding, respectively) involutions

inversion number of permutations

$\mathfrak{S}_n :=$ the set of permutations of $\{1, \ldots, n\}$.

For a $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, the *inversion number* of σ is defined as

$$\mathrm{inv}(\sigma) = |\{(\sigma_i, \sigma_j) : i < j \text{ and } \sigma_i > \sigma_j\}|.$$

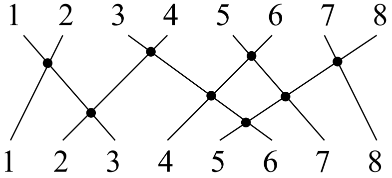
inversion number of permutations

$\mathfrak{S}_n$:= the set of permutations of $\{1, \ldots, n\}$.

For a $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, the *inversion number* of σ is defined as

$$\text{inv}(\sigma) = |\{(\sigma_i, \sigma_j) : i < j \text{ and } \sigma_i > \sigma_j\}|.$$

For example, $\sigma = 31627485 \in \mathfrak{S}_8$, $\text{inv}(\sigma) = 8$.



distribution respecting inversion number

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathsf{inv}(\sigma)} = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$
$$:= [1]_q[2]_q[3]_q\cdots[n]_q,$$

where $[i]_q = 1 + q + \cdots q^{i-1}$ for any positive integer i.

distribution respecting inversion number

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathsf{inv}(\sigma)} = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$
$$:= [1]_q[2]_q[3]_q\cdots[n]_q,$$

where $[i]_q = 1 + q + \cdots q^{i-1}$ for any positive integer i.

A statistic that is equidistributed with inv is called *Mahonian*.

sign-balance of $\mathfrak{S}_n$

The *sign-balance* of $\mathfrak{S}_n$ is 0, i.e.,

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathsf{inv}(\sigma)} = 0.$$

321-avoiding permutations

Let $\mathfrak{S}_n(321)$ be the set of permutations in $\mathfrak{S}_n$ without decreasing subsequences of length three.

321-avoiding permutations

Let $\mathfrak{S}_n(321)$ be the set of permutations in $\mathfrak{S}_n$ without decreasing subsequences of length three.

Simion and Schmidt proved that

$$|\mathfrak{S}_n(321)| = C_n,$$

where $C_n = \frac{1}{2n+1}\binom{2n}{n}$ is the nth Catalan number.

321-avoiding permutations

Let $\mathfrak{S}_n(321)$ be the set of permutations in $\mathfrak{S}_n$ without decreasing subsequences of length three.

Simion and Schmidt proved that

$$|\mathfrak{S}_n(321)| = C_n,$$

where $C_n = \frac{1}{2n+1}\binom{2n}{n}$ is the nth Catalan number.

Theorem (Simion–Schmidt 1985)

$$\sum_{\sigma \in \mathfrak{S}_{2n+1}(321)} (-1)^{\mathit{inv}(\sigma)} = C_n,$$

$$\sum_{\sigma \in \mathfrak{S}_{2n}(321)} (-1)^{\mathit{inv}(\sigma)} = 0.$$

a refined sign-balance result

Let Ides(σ)$= \max\{i : \sigma_i > \sigma_{i+1}, 1 \le i \le n-1\}$,
(Ides$(\sigma) = 0$ for $\sigma = 1\,2\cdots n$).

For example, Ides$(512643) = 5$.

a refined sign-balance result

Let Ides(σ)$= \max\{i : \sigma_i > \sigma_{i+1}, 1 \leq i \leq n-1\}$,
(Ides(σ) $= 0$ for $\sigma = 1\,2 \cdots n$).

For example, Ides$(512643) = 5$.

Theorem (Adin–Roichman 2004)

$$\sum_{\sigma \in \mathfrak{S}_{2n+1}(321)} (-1)^{\mathit{inv}(\sigma)} q^{\mathit{ldes}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n(321)} q^{2 \cdot \mathit{ldes}(\sigma)}.$$

$$\sum_{\sigma \in \mathfrak{S}_{2n}(321)} (-1)^{\mathit{inv}(\sigma)} q^{\mathit{ldes}(\sigma)} = (1-q) \sum_{\sigma \in \mathfrak{S}_n(321)} q^{2 \cdot \mathit{ldes}(\sigma)}.$$

an analogue with respect to lis statistic

Let $\text{lis}(\sigma)$ be the length of longest increasing subsequence in σ.

For example, $\text{lis}(126435) = 4$.

an analogue with respect to lis statistic

Let lis(σ) be the length of longest increasing subsequence in σ.

For example, lis$(126435) = 4$.

Theorem (Reifegerste 2005)

$$\sum_{\sigma \in \mathfrak{S}_{2n+1}(321)} (-1)^{inv(\sigma)} q^{lis(\sigma)} = q \sum_{\sigma \in \mathfrak{S}_{n}(321)} q^{2 \cdot lis(\sigma)}.$$

$$\sum_{\sigma \in \mathfrak{S}_{2n+2}(321)} (-1)^{inv(\sigma)} q^{lis(\sigma)} = q(q-1) \sum_{\sigma \in \mathfrak{S}_{n}(321)} q^{2 \cdot lis(\sigma)}.$$

a phenomenon (up to small variations)

The signed enumerator of objects of size $2n$ is essentially equal to the ordinary enumerator of objects of size n.

$$\sum_{\pi \in \mathcal{X}_{2n} \text{ or } \mathcal{X}_{2n+1}} (-1)^{\mathsf{stat}_1(\pi)} q^{\mathsf{stat}_2(\pi)} = f(q) \sum_{\pi \in \mathcal{X}_n} q^{2 \cdot \mathsf{stat}_2(\pi)},$$

where $\mathcal{X}_n$ is a family of objects of size n with statistics stat_1 and stat_2, and $f(q)$ is a rational function.

alternating permutations

A permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ is alternating if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$.

alternating permutations

A permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ is alternating if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$.

Let $\mathsf{Alt}_n(321)$ be the set of 321-avoiding alternating permutations in $\mathfrak{S}_n$.

Theorem (Deutsch–Reifegerste 2003; Mansour 2003)

$$|Alt_{2n}(321)| = |Alt_{2n-1}(321)| = C_n.$$

a refined sign-balance on $Alt_n(321)$

Let lead(σ) denote the first entry of σ. Let $\mathcal{A}_n = \text{Alt}_n(321)$.

Theorem (Eu–Fu–Pan–Ting 2012)

$$\sum_{\sigma\in\mathcal{A}_{4n+2}} (-1)^{inv(\sigma)} q^{lead(\sigma)} = (-1)^{n+1} \sum_{\sigma\in\mathcal{A}_{2n}} q^{2lead(\sigma)}$$

$$\sum_{\sigma\in\mathcal{A}_{4n+1}} (-1)^{inv(\sigma)} q^{lead(\sigma)} = (-1)^{n} \sum_{\sigma\in\mathcal{A}_{2n}} q^{2lead(\sigma)}$$

$$\sum_{\sigma\in\mathcal{A}_{4n}} (-1)^{inv\sigma)} q^{lead(\sigma)} = (-1)^{n+1}(1-q) \sum_{\sigma\in\mathcal{A}_{2n}} q^{2(lead(\sigma)-1)}$$

$$\sum_{\sigma\in\mathcal{A}_{4n-1}} (-1)^{inv(\sigma)} q^{lead(\sigma)} = (-1)^{n}(1-q) \sum_{\sigma\in\mathcal{A}_{2n}} q^{2(lead(\sigma)-1)}$$

descent number and major index

$\sigma \in \mathfrak{S}_n$, let Des($\sigma$)$= \{i : \sigma_i > \sigma_{i+1}, 1 \le i \le n-1\}$. The *descent number* (des) and *major index* (maj) of σ are defined by

$$\mathrm{des}(\sigma) = |\mathrm{Des}(\sigma)| \quad \text{and} \quad \mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i.$$

descent number and major index

$\sigma \in \mathfrak{S}_n$, let Des($\sigma$)$= \{i : \sigma_i > \sigma_{i+1}, 1 \leq i \leq n-1\}$. The *descent number* (des) and *major index* (maj) of σ are defined by

$$\mathrm{des}(\sigma) = |\mathrm{Des}(\sigma)| \quad \text{and} \quad \mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i.$$

For $\sigma = 7256314$, $\mathrm{Des}(\sigma) = \{1, 4, 5\}$, $\mathrm{des}(\sigma) = 3$, $\mathrm{maj}(\sigma) = 10$.

The major index is Mahonian, i.e.,

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}.$$

involutions

$\sigma \in \mathfrak{S}_n$ is an involution if and only if $\sigma^{-1} = \sigma$. Let $\mathcal{I}_n(321)$ ($\mathcal{I}_n(123)$, respectively) be the set of 321-avoiding (123-avoiding, respectively) involutions in $\mathfrak{S}_n$.

involutions

$\sigma \in \mathfrak{S}_n$ is an involution if and only if $\sigma^{-1} = \sigma$. Let $\mathcal{I}_n(321)$ ($\mathcal{I}_n(123)$, respectively) be the set of 321-avoiding (123-avoiding, respectively) involutions in $\mathfrak{S}_n$.

Theorem (Simion–Schmidt 1985)

$$|\mathcal{I}_n(321)| = |\mathcal{I}_n(123)| = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

a major-balance result on 321-avoding involutions

Theorem (Eu–Fu–Pan–Ting 2015)

For $n \geq 0$, we have

$$\sum_{\sigma \in \mathcal{I}_{4n}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{des}(\sigma)} = \sum_{\sigma \in \mathcal{I}_{2n}(321)} q^{2 \cdot \mathsf{des}(\sigma)}.$$

$$\sum_{\sigma \in \mathcal{I}_{4n+2}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{des}(\sigma)} = (1-q) \sum_{\sigma \in \mathcal{I}_{2n}(321)} q^{2 \cdot \mathsf{des}(\sigma)},$$

$$\sum_{\sigma \in \mathcal{I}_{2n+1}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{des}(\sigma)} = \sum_{\sigma \in \mathcal{I}_{n}(321)} q^{2 \cdot \mathsf{des}(\sigma)}.$$

an analogue with respect to fixed point statistic

$\sigma \in \mathfrak{S}_n$, let fix($\sigma$)$= \{i : \sigma_i = i, 1 \leq i \leq n\}$.

an analogue with respect to fixed point statistic

$\sigma \in \mathfrak{S}_n$, let fix($\sigma$)$= \{i : \sigma_i = i, 1 \le i \le n\}$.

Theorem (Eu–Fu–Pan–Ting 2015)

For $n \ge 0$, *we have*

$$\sum_{\sigma \in \mathcal{I}_{2n+1}(321)} (-1)^{\mathit{maj}(\sigma)} q^{\mathit{fix}(\sigma)} = q \sum_{\sigma \in \mathcal{I}_n(321)} q^{2 \cdot \mathit{fix}(\sigma)}.$$

a major-balance result on 123-avoiding involutions

Theorem (Fu–Hsu–Liao 2018)

For $n \geq 1$, we have

$$\sum_{\sigma \in \mathcal{I}_{4n}(123)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{des}(\sigma)} = q \sum_{\sigma \in \mathcal{I}_{2n}(123)} q^{2 \cdot \mathsf{des}(\sigma)}.$$

$$\sum_{\sigma \in \mathcal{I}_{4n+2}(123)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{des}(\sigma)} = (1-q) q^2 \sum_{\sigma \in \mathcal{I}_{2n}(123)} q^{2 \cdot \mathsf{des}(\sigma)},$$

$$\sum_{\sigma \in \mathcal{I}_{2n+1}(123)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{des}(\sigma)} = (-1)^n q^2 \sum_{\sigma \in \mathcal{I}_{n}(123)} q^{2 \cdot \mathsf{des}(\sigma)}.$$

an analogous result with respect to leading element

Theorem (Fu–Hsu–Liao 2018)

For $n \geq 1$, we have

$$\sum_{\sigma \in \mathcal{I}_{4n}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{lead}(\sigma)} = \frac{1}{q} \sum_{\sigma \in \mathcal{I}_{2n}(321)} q^{2 \cdot \mathsf{lead}(\sigma)}.$$

$$\sum_{\sigma \in \mathcal{I}_{4n+2}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{lead}(\sigma)} = \left(\frac{1}{q} - 1\right) \sum_{\sigma \in \mathcal{I}_{2n+1}(321)} q^{2 \cdot \mathsf{lead}(\sigma)},$$

$$\sum_{\sigma \in \mathcal{I}_{4n+3}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{lead}(\sigma)} = \left(\frac{2}{q} - 1\right) \sum_{\sigma \in \mathcal{I}_{2n+1}(321)} q^{2 \cdot \mathsf{lead}(\sigma)},$$

$$\sum_{\sigma \in \mathcal{I}_{4n+1}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{des}(\sigma)} = \left(\frac{1}{q} - 1\right) \sum_{\sigma \in \mathcal{I}_{2n+1}(321)} q^{2 \cdot \mathsf{des}(\sigma)} + \sum_{\sigma \in \mathcal{I}_{2n}(321)} q^{2 \cdot \mathsf{lead}(\sigma)}.$$

a combinatorial proof: lattice paths

Let $\mathcal{B}_n$ be the set of lattice paths π from $(0,0)$ to $(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$.

Barnabei–Bonetti–Elizalde–Silimbani (2014) established a bijection $\xi : \sigma \mapsto \pi$ of $\mathcal{I}_n(321)$ onto $\mathcal{B}_n$ such that

(i) $\mathsf{maj}(\sigma) = \mathsf{sump}(\pi)$, the sum of the x-coordinates and y-coordinates of all peaks in π.

(ii) $\mathsf{lead}(\sigma) - 1$ equals the height of the first peak of π.

a combinatorial proof: lattice paths

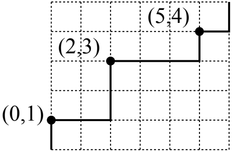
Let $\mathcal{B}_n$ be the set of lattice paths π from $(0,0)$ to $(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$.

Barnabei–Bonetti–Elizalde–Silimbani (2014) established a bijection $\xi : \sigma \mapsto \pi$ of $\mathcal{I}_n(321)$ onto $\mathcal{B}_n$ such that

(i) maj$(\sigma) =$ sump(π), the sum of the x-coordinates and y-coordinates of all peaks in π.

(ii) lead$(\sigma) - 1$ equals the height of the first peak of π.

For example, the corresponding path of $\sigma = 2\,1\,3\,6\,7\,4\,5\,8\,10\,9\,11 \in \mathcal{I}_{11}(321)$ with lead$(\sigma) = 2$ and maj$(\sigma) = 15$ is shown below.



a sign-reversing involution on lattice paths

To prove

$$\sum_{\sigma \in \mathcal{I}_{4n}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{lead}(\sigma)} = \frac{1}{q} \sum_{\sigma \in \mathcal{I}_{2n}(321)} q^{2 \cdot \mathsf{lead}(\sigma)},$$

we shall establish a sump-parity-reversing involution $\Phi_1 : \mathcal{B}(2n, 2n) \to \mathcal{B}(2n, 2n)$ while preserving the height of the first peak of the lattice paths.

a sign-reversing involution on lattice paths

To prove

$$\sum_{\sigma \in \mathcal{I}_{4n}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{lead}(\sigma)} = \frac{1}{q} \sum_{\sigma \in \mathcal{I}_{2n}(321)} q^{2 \cdot \mathsf{lead}(\sigma)},$$

we shall establish a sump-parity-reversing involution $\Phi_1 : \mathcal{B}(2n, 2n) \to \mathcal{B}(2n, 2n)$ while preserving the height of the first peak of the lattice paths.

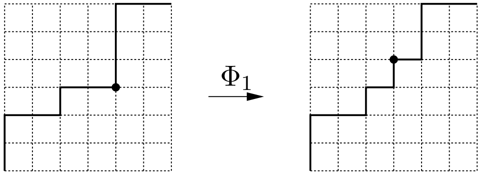
For any lattice path $\pi \in \mathcal{B}(2n, 2n)$, we factorize π as $\pi = \mu_0 \mu_1 \cdots \mu_d$, where each segment μ_{2i} (μ_{2i+1}, respectively) is a maximal sequence of consecutive north steps (east steps, respectively).

the construction of the map Φ_1

Algorithm: If every segment μ_i contains an even number of steps then $\Phi_1(\pi) = \pi$. Otherwise, find the greatest integer k such that μ_k contains an odd number of steps. The path $\Phi_1(\pi)$ is obtained from π by interchanging the first step of μ_k and the last step of μ_{k-1}.

the construction of the map Φ_1

Algorithm: If every segment μ_i contains an even number of steps then $\Phi_1(\pi) = \pi$. Otherwise, find the greatest integer k such that μ_k contains an odd number of steps. The path $\Phi_1(\pi)$ is obtained from π by interchanging the first step of μ_k and the last step of μ_{k-1}.



a sign-reversing involution on lattice paths

To prove

$$\sum_{\sigma \in \mathcal{I}_{4n+2}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{lead}(\sigma)} = \left(\frac{1}{q} - 1\right) \sum_{\sigma \in \mathcal{I}_{2n+1}(321)} q^{2\cdot \mathsf{lead}(\sigma)},$$

we shall establish a sump-parity-reversing involution $\Phi_2 : \mathcal{B}(2n+1, 2n+1) \to \mathcal{B}(2n+1, 2n+1)$ while preserving the height of the first peak of the lattice paths.

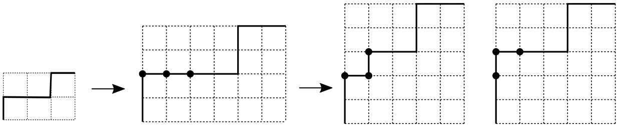
a sign-reversing involution on lattice paths

To prove

$$\sum_{\sigma \in \mathcal{I}_{4n+2}(321)} (-1)^{\mathsf{maj}(\sigma)} q^{\mathsf{lead}(\sigma)} = \left(\frac{1}{q} - 1\right) \sum_{\sigma \in \mathcal{I}_{2n+1}(321)} q^{2 \cdot \mathsf{lead}(\sigma)},$$

we shall establish a sump-parity-reversing involution $\Phi_2 : \mathcal{B}(2n+1, 2n+1) \to \mathcal{B}(2n+1, 2n+1)$ while preserving the height of the first peak of the lattice paths.

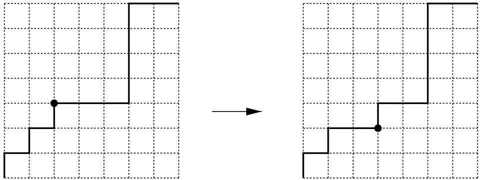
Construction of fixed points of the map Φ_2:



the construction of the map Φ_2

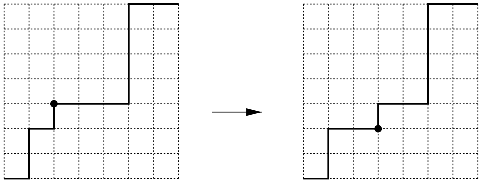
Algorithm: Given a path $\pi \in \mathcal{B}(2n+1, 2n+1)$ with the factorization $\mu_0\mu_1 \cdots \mu_d$, there are two cases.

Case 1. μ_0 is of odd length, say $\mu_0 = \mathsf{N}^{2j+1}$. Find the greatest integer $k \geq 1$ such that μ_k contains an odd number of steps. If $k = 1$ then let $\Phi_2(\pi) = \pi$. Otherwise, the path $\Phi_1(\pi)$ is obtained from π by interchanging the first step of μ_k and the last step of μ_{k-1}.



the construction of the map Φ_2

Case 2. μ_0 is of even length, say $\mu_0 = \mathsf{N}^{2j}$. If $\mu_1 = \mathsf{E}$ and μ_2 is the only other segment containing an odd number of steps (i.e., μ_t is of even length for all $t \geq 3$) then let $\Phi_2(\pi) = \pi$. Otherwise, find the greatest integer $k \geq 3$ such that μ_k contains an odd number of steps. Then the path $\Phi_2(\pi)$ is obtained from π by interchanging the first step of μ_k and the last step of μ_{k-1}.



references

- R.M. Adin, Y. Roichman, Equidistribution and sign-balance on 321-avoiding permutations, Sémin. Loth. Combin. 51 (2004) B51d.
- M. Barnabei, F. Bonetti, S. Elizalde, M. Silimbani, Descent sets on 321-avoiding involutions and hook decompositions of partitions, J. Combin. Theory Ser. A 128 (2014) 132–148.
- A. Reifegerste, Refined sign-balance on 321-avoiding permutations, European J. Combin. 26 (2005) 1009–1018.
- R. Simion, F.W. Schmidt, Restricted permutations, European J. Combin. 6 (1985) 383–406.

Thanks for your attention.