Hamiltonicity of edge-chromatic critical graphs

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Notations and Definitions

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For a graph G and $S \subseteq V(G)$, if for any two distinct vertices $u, v \in S$, $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$, then S is *elementary* with respect to φ, where φ is an edge coloring of G.

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[Holyer 1981] It is NP-complete to determine whether a graph is Class I or Class II.

[1]. V. G. Vizing. Critical graphs with a given chromatic index (in russian). Diskret. Analiz No., 5:9-17, 1965.

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  - $d(x) + d(y) \geq \Delta + 2$ for $xy \in E(G)$;
  - if $d(x) + d(y) = \Delta + 2$, then every neighbor of x and y, other than (possibly) x and y themselves, has degree Δ.

Vizing's 2-Factor Conjecture

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[4] G. Chen and S. Shan. Vizing's 2-factor conjecture involving large maximum degree. J. Graph Theory, 00:1-17, 2017.

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Obviously, if a graph is Hamiltonian, then it contains a 2-factor. By increasing maximum degrees in terms the orders of graphs, Luo and Zhao [7] proved that the following result.

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[7] R. Luo and Y. Zhao. A sufficient condition for edge chromatic critical graphs to be Hamiltonian. J. Graph Theory, 73(4): 469-482, 2013.

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- We show that an edge-Δ-critical graph G of order n with $\Delta \geq \frac{2n}{3} + 12$ is Hamiltonian.

[8] R. Luo, Z. Miao, and Y. Zhao. Hamiltonian cycles in critical graphs with large maximum degree. Graphs Combin., 32(5): 2019-2028, 2016.

[9] G. Chen, X. Chen, and Y. Zhao. Hamiltonianicity of edge chromatic critical graph. Discrete Math. 340: 3011-3015, 2017.

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A *multi-fan at x* with respect to e and φ is a sequence $F = (e_1, y_1, ..., e_p, y_p)$ with $p > 1$ consisting of edges $e_1, ..., e_p$ and vertices $y_1, ..., y_p$ satisfying the following two conditions:

- $e_1, ..., e_p$ are distinct, $e_i = e$, and $e_i = xy_i$ for $i = 1, ..., p$.
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Theorem

$V(F)$ is elementary with respect to φ.

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Theorem (Kierstead, JCTB, 1984)

If $d(y_i) < \Delta$ for $i = 2, ..., p$, then $V(K)$ is elementary w.r.t. ϕ.

Conjecture (G Chen)

Let K be a Kierstead path, then there are at most $f(n, \Delta)$ pairs of missing colors repeated.

Kierstead Paths with Four Vertices

Kostochka and Stiebitz [11] considered elementary property of Kierstead paths with four vertices and showed the following:

Lemma (*Kostochka, Stiebitz [11]*)

Let G be a graph with maximum degree Δ and $\chi'(G) = \Delta + 1$. Let $e_1 \in E(G)$ be a critical edge and $\varphi \in \mathcal{C}^{\Delta}(G - e_1)$. If $K = (y_0, e_1, y_1, e_2, y_2, e_3, y_3)$ is a Kierstead path with respect to e_1 and φ, then the following statements hold:

1. $\bar{\varphi}(y_0) \cap \bar{\varphi}(y_1) = \emptyset$;
2. *if $d(y_2) < \Delta$, then $V(K)$ is elementary with respect to φ;*
3. *if $d(y_1) < \Delta$, then $V(K)$ is elementary with respect to φ;*
4. *if $\Gamma = \bar{\varphi}(y_0) \cup \bar{\varphi}(y_1)$, then $|\bar{\varphi}(y_3) \cap \Gamma| \leq 1$.*

[11] M. Stiebitz, D. Scheide, B. Toft, and L. M. Favrholdt. Graph edge-coloring: Vizing's theorem and Goldberg's conjecture. Wiley, 2012.

Broom

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We call $B = \{y_0, y_1, y_2\} \cup Z$ is a *broom* if for every vertex $z \in Z$, $(y_0, e_1, y_1, e_2, y_2, e_3, z)$ is a Kierstead path with respect to e_1 and φ.

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Theorem (Chen, Chen and Zhao [8])

Let G be an edge-Δ-critical graph, $e_1 = y_0 y_1 \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G - e_1)$ and $B = \{y_0, y_1, y_2\} \cup Z$ be a broom. If $|\bar{\varphi}(y_0) \cup \bar{\varphi}(y_1)| \geq 4$ and $\min\{d(y_1), d(y_2)\} < \Delta$, then $V(B)$ is elementary with respect to φ.

Vizing's Adjacency Lemma

Let q be a positive number, G be an edge-Δ-critical graph and $x \in V(G)$.

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For each $y \in N(x)$, let $\sigma_q(x, y) = |\{z \in N(y) \setminus \{x\} \; : \; d(z) \geq q\}|$, the number of neighbors of y (except x) with degree at least q.

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Lemma (*Vizing's Adjacency Lemma*)

Let G be an edge-Δ-critical graph. Then $\sigma_\Delta(x, y) \geq \Delta - d(x) + 1$ holds for every $xy \in E(G)$.

Woodall's result I

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Theorem (Woodall, 2007)

Let xy be an edge in an edge-Δ-critical graph G. Then there are at least $\Delta - \sigma(x, y) \geq \Delta - d(y) + 1$ vertices $z \in N(x) \setminus \{y\}$ such that

$$\sigma(x, z) + \sigma(x, y) \geq 2\Delta - d(x).$$

[12] D. R. Woodall. The average degree of an edge-chromatic critical graph II. J. Graph Theory, 56(3):194-218, 2007.

Woodall's result II

Furthermore, Woodall defined the following two parameters.

$$p_{min}(x) := \min_{y \in N(x)} \sigma(x,y) - \Delta + d(x) - 1 \quad \text{and}$$

$$p(x) := \min\{\ p_{min}(x), \left\lfloor \frac{d(x)}{2} \right\rfloor - 1\ \}.$$

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Theorem (Woodall, 2007)

Every vertex x in an edge-Δ-critical graph has at least $d(x) - p(x) - 1$ neighbors y for which $\sigma(x,y) \geq \Delta - p(x) - 1$.

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- $p(x) < d(x)/2 - 1$.
- $\exists$ about $d(x)/2$ neighbors y of x s.t. $\sigma(x,y)$ is at least $\Delta/2$.

General Vizing's Fan

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Lemma

Let xy be an edge in a Δ-critical graph G and q be a positive number. If $d(x) < \frac{\Delta(G)}{2}$ and $q \le \Delta(G) - 10$, then there exists a vertex $z \in N(x) \setminus \{y\}$ such that

$$\begin{aligned} & \sigma_q(x, y) + \sigma_q(x, z) \\ > \; & 2\Delta(G) - d(x) - \frac{2d(x) - 2}{\Delta(G) - q} - \left\lceil \frac{4d(x) - 4}{\Delta(G) - q} + \frac{8(d(x) - 1)}{(\Delta(G) - q)^2} \right\rceil . \end{aligned}$$

General Broom

Lemma

Let x_1x_2 be an edge in a Δ-critical graph G and q be a positive number. If $t = d(x_1) + d(x_2) \leq \frac{3}{2}\Delta(G) - 2$, $q \leq \Delta(G) - 10$ and $\delta(G) > \frac{\Delta(G)}{2} - 2$, then there exists a pair of vertices $\{z, y\}$ with $z \in N(x_1) \setminus \{x_2\}$ and $y \in N(x_2) \setminus \{x_1, z\}$ such that

$$\begin{aligned} & \sigma_q(x_1, z) + \sigma_q(x_2, y) \\ > \quad & 3\Delta(G) - t - \frac{2(t - \Delta(G) - 2)}{\Delta(G) - q} - 2 \\ & - \left[\frac{4(t - \Delta(G) + 2)}{\Delta(G) - q} + \frac{8(t - \Delta(G) - 2)}{(\Delta(G) - q)^2} \right]. \end{aligned}$$

Our Result

Theorem

If G is an edge-Δ-critical graph of order n with $\Delta \geq \frac{2n}{3} + 12$, then G is Hamiltonian.

Lemmas

Chen, Chen and Zhao showed the following Lemma.

Lemma (Chen, Chen, Zhao [9], 2017)

Let G be an edge-Δ-critical graph of order n. If $d(x) + d(y) \geq n$ for any edge xy of G, then G is Hamiltonian.

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Brandt and Veldman gave the following result about the circumference of a graph G and its closure $C(G)$.

Lemma

A graph G has the same circumference as its closure $C(G)$.

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- In Claims 3.1, 3.3 and 3.4 we need the inequalities $n < \frac{3}{2}\Delta - 1$, $r + \Delta \geq n - 1$, and $r < \frac{\Delta}{2} + 1$, respectively. Thus we need $\Delta \geq \frac{2}{3}n + k$, for some k.

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- For convenience we will choose $r = \frac{\Delta}{2} - 2$, and the proof (specifically, Claim 3.2) works with $k = 12$.
- The numbers appearing in Claim 3.2 are related to k by $18 = \frac{3}{2}k$, $\frac{757}{1156} = \frac{3}{4} - \frac{1}{k} - \frac{8}{9k^2}$, and $\frac{179}{1156} = \frac{757}{1156} - \frac{1}{2}$; this last equation is used in Claim 3.3.

Sketch of Proof II

Claim (3.1)

Suppose q is a positive number and $q \leq \Delta - 10$. *Then*

$$|V_{\geq q}(G)| > \frac{3\Delta}{4} - \frac{3\Delta}{2(\Delta - q)} - \frac{2\Delta}{(\Delta - q)^2}; \tag{1}$$

if $\delta(G) \leq \frac{\Delta}{2} - 2$, *then*

$$|V_{\geq q}(G)| > \frac{3\Delta}{4} - \frac{3\Delta - 18}{2(\Delta - q)} - \frac{2\Delta - 12}{(\Delta - q)^2} + \frac{1}{2}; \tag{2}$$

and if $\delta(G) > \frac{\Delta}{2} - 2$, *then*

$$|V_{\geq q}(G)| > \frac{3\Delta}{4} - \frac{3\Delta - 110}{2(\Delta - q)} - \frac{2\Delta - 84}{(\Delta - q)^2} + 8. \tag{3}$$

Sketch of Proof III

Claim (3.2)

The following inequalities hold:

a. $|V_{\geq \Delta-17}(G)| > \frac{757}{1156}\Delta$;

b. $|V_{\geq \Delta-17}(G)| > \frac{n}{2}$ *provided* $\Delta \leq 94$;

c. $|V_{\geq (1-\frac{179}{1156})\Delta}(G)| \geq \frac{n}{2}$ *provided* $\Delta \geq 95$.

Claim (3.3)

In $C(G)$, $V_{\geq \frac{\Delta}{2}}(G)$ *is a clique.*

So $C(G)$ is Hamiltonian, and G is Hamiltonian, a contradiction.

Thank You For Your Attention!