Clustered coloring for old graph coloring conjectures

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Graph coloring

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Answer: No!

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More structures?

A graph is a *planar graph* if it can be drawn in the plane without edge-crossing.

Four Color Theorem: Every planar graph is properly 4-colorable.

Hadwiger's conjecture

A graph G contains another graph H as a *minor* if H can be obtained from a subgraph of G by repeatedly contracting edges.

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Hadwiger's conjecture: Every K_{t+1}-minor free graph is properly t-colorable.

- **(Hadwiger):** True for $t \leq 3$.
- **(Wagner):** Equivalent with the Four Color Theorem for $t = 4$.
- **(Robertson, Seymour, Thomas):** True for $t = 5$.
- Open for $t \geq 6$.
- **(Kostochka; Thomason:)** $O(t\sqrt{\log t})$ colors suffice.

Topological minors

A graph G contains another graph H as a *topological minor* if H can be obtained from a subgraph of G by repeatedly contracting edges incident with vertices of degree two.

Hajós' conjecture: Every K_{t+1}-topological minor free graph is properly t-colorable.

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Hajós' conjecture: Every K_{t+1}-topological minor free graph is properly t-colorable.

- **(Dirac):** True for $t \leq 3$.
- **(Catlin):** False for $t \geq 6$.
- Open for $t \in \{4, 5\}$.
- **(Erdős, Fajtlowicz):** Need $\Omega(t^2/\log t)$ colors.
- **(Bollobás, Thomason; Komlós, Szemerédi):** $O(t^2)$ colors suffice.

Odd minors

A graph G contains another graph H as an *odd minor* if H can be obtained from a subgraph of G by contracting all edges in an edge-cut.

Gerards-Seymour Conjecture: Every odd K_{t+1}-minor free graph is properly t-colorable.

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Gerards-Seymour Conjecture: Every odd K_{t+1}-minor free graph is properly t-colorable.

- **(Catlin):** True for $t \leq 3$.
- **(Guenin):** True for $t \leq 4$.
- Open for $t \geq 5$.
- **(Geelen, Gerards, Reed, Seymour, Vetta):** $O(t\sqrt{\log t})$ colors suffice.

Clustered coloring

A graph G is properly k-colorable if and only if the vertices of G can be colored with k colors such that every monochromatic component contains only 1 vertex.

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A class $\mathcal{F}$ of graphs is *clustered k-colorable* if there exists an integer N such that for every $G \in \mathcal{F}$, the vertices of G can be colored with k colors such that every monochromatic component contains at most N vertices.

Standard construction

Let $\mathcal{F}_{k,N}$ be the set of graphs such that every k-coloring leads to a monochromatic component on more than N vertices.

A class $\mathcal{F}$ of graph is clustered k-colorable $\Leftrightarrow$ there exists N such that $\mathcal{F} \cap \mathcal{F}_{k,n} = \emptyset$ for every $n \geq N$.

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Standard construction:

- Let $H \in \mathcal{F}_{k,N}$.
- Let G be the graph obtained from a union of N disjoint copies of H by adding a new vertex v adjacent to all other vertices.
- Then $G \in \mathcal{F}_{k+1,N}$.

Hex Lemma: Any 2-coloring of the $N \times N$ triangular grid leads to a monochromatic path on N vertices.

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Corollary: Let $q \geq 6$. If $\mathcal{F}$ is a set of graphs of maximum degree at most q and contains all planar graphs, then $\mathcal{F}$ is not clustered 2-colorable.

Clustered coloring and planar graphs

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Theorem: (Alon, Ding, Oporowski, Vertigan) For any q and any planar graph H, the class of H-minor-free graphs of maximum degree at most q is clustered 2-colorable.

Minor closed families of bounded maximum degree graphs

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3. **(Esperet, Joret):** For any q and any surface Σ, the class of graphs of maximum degree at most q embeddable in Σ is clustered 3-colorable.

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3. **(Esperet, Joret):** For any q and any surface Σ, the class of graphs of maximum degree at most q embeddable in Σ is clustered 3-colorable.
4. **(L., Oum):** For any q and any graph H, the class of odd H-minor-free graphs of maximum degree at most q is clustered 3-colorable.

Clustered list-coloring

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A class $\mathcal{F}$ of graphs is clustered k-choosable if there exists an integer N such that for every list-assignment $L = (L_v : v \in V(G))$ of any graph $G \in \mathcal{F}$ with $|L(v)| \geq k$ for every $v \in V(G)$, there exists a coloring f such that $f(v) \in L_v$ for every $v \in V(G)$ such that every monochromatic component contains at most N vertices.

Theorem: (L.) For any q and any planar graph H, the class of H-minor-free graphs of maximum degree at most q is clustered 2-choosable.

Tree-decomposition

$(T, \mathcal{X})$ is a *tree-decomposition* of G if the following hold.

- T is a tree, and $\mathcal{X} = \{X_t : t \in V(T)\}$, where each X_t is a subset of $V(G)$ and $\bigcup_{t \in V(T)} X_t = V(G)$.
- For every edge of G, its both ends are in some X_t.
- For every vertex v of G, the subgraph of T induced by $\{t \in V(T) : v \in X_t\}$ is connected.

The *width* of $(T, \mathcal{X})$ is $\max_{t \in V(T)} |X_t| - 1$.

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The *tree-width* of a graph is the minimum width of its tree-decompositions.

Theorem (L.): For any q, w, the class of graphs of maximum degree at most q and tree-width most w is clustered 2-choosable.

Theorem: (Esperet, Joret) For any q and any surface Σ, the class of graphs of maximum degree at most q embeddable in Σ is clustered 3-colorable.

Layered tree-width

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A *layering* of a graph G is an ordered partition $(V_1, V_2, ...)$ of $V(G)$ such that for every edge e of G, there exists i such that either both ends of e are contained in V_i, or e is between V_i and V_{i+1}.

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The *layered tree-width* of G is the minimum w such that there exist a layering $(V_1, V_2, \ldots)$ and a tree-decomposition $(T, \mathcal{X})$ such that $\max_{i \in \mathbb{N}, t \in V(T)} |V_i \cap X_t| = w$.

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Theorem: (Dujmović, Morin, Wood) Every graph embeddable in a surface of Euler genus g has layered tree-width at most $2g + 3$.

Theorem: (Esperet, Joret) For any q and any surface Σ, the class of graphs of maximum degree at most q embeddable in Σ is clustered 3-colorable such that every monochromatic component contains $O(q^{32q2^g})$ vertices.

Theorem (L., Wood)

For any q, w, the class of graphs of maximum degree at most q and layered tree-width at most w is clustered 3-colorable such that every monochromatic component contains $O(w^{22}q^{43})$ vertices.

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- Classes of graphs of bounded layered treewidth are not minor closed.
- K_6-minor-free graphs can have arbitrarily large layered treewidth.
- Graphs of layered treewidth 2 can contain arbitrarily large graph as a minor.

Summary for graphs of bounded maximum degree

Theorem:

1. **(L.):** For any q, w, the class of graphs of maximum degree at most q and tree-width at most w is clustered 2-choosable.
2. **(L., Wood):** For any q, w, the class of graphs of maximum degree at most q and layered tree-width at most w is clustered 3-colorable.
3. **(L., Oum):** For any q and graph H, the class of H-minor-free graphs of maximum degree at most q is clustered 3-colorable.
4. **(L., Oum):** For any q and graph H, the class of odd H-minor-free graphs of maximum degree at most q is clustered 3-colorable.

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Maximum degree at most $q \Leftrightarrow$ no $K_{1,q+1}$-subgraph.

Excluding complete bipartite subgraphs

Theorem (L., Wood)

1. *For any p, q, w, the class of graphs of tree-width at most w with no $K_{p,q}$-subgraph is clustered $p+1$-choosable.*
2. *For any p, q, w, the class of graphs of layered tree-width at most w with no $K_{p,q}$-subgraph is clustered $p+2$-colorable.*
3. *For any p, q and graph H, the class of H-minor-free graphs with no $K_{p,q}$-subgraph is clustered $p+2$-colorable.*
4. *For any p, q and graph H, the class of odd H-minor-free graphs with no $K_{p,q}$-subgraph is clustered $2p+1$-colorable.*
5. *For any p, q, d with $p+3d \geq 7$ and graph H of maximum degree d, the class of H-topological minor free graphs with no $K_{p,q}$-subgraph is clustered $(p+3d-4)$-colorable.*

Statements 1 and 3 are tight for every p.

Application to Hadwiger's conjecture

Question: For every t, what is the minimum $f(t)$ such that the class of K_{t+1}-minor-free graphs is clustered $f(t)$-colorable?

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Theorem:

- (Edwards, Kang, Kim, Oum, Seymour) $t \leq f(t) \leq 4t$.
- (Kawarabayashi, Mohar) $f(t) \leq \lceil \frac{31}{2}(t+1) \rceil$.
- (Wood) $f(t) \leq \lceil \frac{7}{2}t + 2 \rceil$.
- (L., Oum) $f(t) \leq 3t$.
- (Norin; van den Heuval, Wood) $f(t) \leq 2t$.

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No K_{t+1}-minor $\Rightarrow$ no $K_{t,t}$-subgraph.

Theorem (L., Wood)

$f(t) \leq t + 2$.

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Theorem: (Dvořák, Norin) $f(t) = t$.

Application to Gerards-Seymour Conjecture

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- (Kawarabayashi) $f(t) \leq 496t$.
- (Kang, Oum) $f(t) \leq 10t - 13$.

Theorem (L., Wood)

$f(t) \leq 8t - 4$.

Application to Hajós' conjecture

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$t \leq f(t) = O(t^2)$.

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Theorem (L., Wood)

$f(t) \leq 4t - 5$ *if* $t \geq 2$.

Application Hajós' conjecture

No K_{t+1}-topological minor $\Rightarrow$ no $K_{t,\binom{t}{2}+1}$-subgraph.

Theorem (L., Wood)

1. *For any t, w, the class of K_{t+1}-topological minor free graphs of tree-width at most w is clustered $t+1$-choosable.*
2. *For any t, w, the class of K_{t+1}-topological minor free graphs of layered tree-width at most w is clustered $t+2$-colorable.*
3. *For any t and graph H, the class of K_{t+1}-topological minor and H-minor free graphs is clustered $t+2$-colorable.*
4. *For any t and graph H, the class of K_{t+1}-topological minor and odd H-minor free graphs is clustered $2t+1$-colorable.*
5. *For any t, d and graph H of maximum degree d, the class of K_{t+1}-topological minor and H-topological minor free graphs is clustered $(t+3d-4)$-colorable.*

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Application to embedded graphs with crossings

A graph is *(g, k)-planar* if it can be drawn in a surface of Euler genus at most g such that there are at most k edge-crossings on each edge.

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Theorem:

1. **(Esperet, Ochem)** The class of $(g,0)$-planar graphs is clustered 5-colorable.
2. **(Wood)** The class of (g,k)-planar graphs is clustered 12-colorable.

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2. **(Wood)** The class of (g, k)-planar graphs is clustered 12-colorable.
3. **(Dujmović, Eppstein, Wood)** Every (g, k)-planar graph has layered tree-width at most $O(gk)$.
4. **(Ossona de Mendez, Oum, Wood)** Every (g, k)-planar graph has no $K_{3,t}$ subgraph for some large $t \leq O(kg^2)$.
5. **(L., Wood)** For any p, q, w, the class of graphs of layered tree-width at most w with no $K_{p,q}$-subgraph is clustered $p + 2$-colorable.

Application to embedded graphs with crossings

A graph is *(g, k)-planar* if it can be drawn in a surface of Euler genus at most g such that there are at most k edge-crossings on each edge.

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5. **(L., Wood)** For any p, q, w, the class of graphs of layered tree-width at most w with no $K_{p,q}$-subgraph is clustered $p + 2$-colorable.

Corollary (L., Wood)

The class of (g, k)-planar graph is clustered 5-colorable.

Application to geometric graphs

A graph G is a *(g, d)-map graph* if there exist a graph G_0 embedded in a surface of Euler genus at most g with no edge-crossing and a partition of $F(G_0)$ into two parts X_1, X_2 such that

- every vertex of G_0 is incident with at most d faces in X_1, and
- $V(G) = X_1$, and two vertices in G are adjacent if and only if they share a vertex of G_0.

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The class of $(g,3)$-map graphs equals the class of graphs of Euler genus at most g.

Theorem: (Dujmović, Eppstein, Wood) Every (g,d)-map graph is $(g, O(d^2))$-planar.

Corollary (L., Wood)

The class of (g,d)-map graphs is clustered 5-colorable.

Application to geometric graphs

A *(g,k)-string graph* is the intersection graph of a set of curves on a surface of Euler genus at most g, where every curve intersects at most k other curves.

Theorem: (Dujmović, Joret, Morin, Norin, Wood) Every (g,k)-string graph has layered treewidth at most $O(gk)$.

Corollary (L., Wood)

The class of (g,k)-string graphs is clustered 3-colorable.

Future work

Question:

1. Is the class of K_{t+1}-topological minor free graphs clustered t-colorable?
2. Is the class of odd K_{t+1}-minor free graphs clustered t-colorable?
3. Is the class of K_{t+1}-topological minor free graphs clustered t-choosable?

Future work

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3. Is the class of K_{t+1}-topological minor free graphs clustered t-choosable?
4. What is the minimum $f(d)$ such that the class of graphs of maximum degree at most d is clustered $f(d)$-colorable?
    - **(Alon, Ding, Oporowski, Vertigan; Haxell, Szabó, Tardos)** $\lfloor \frac{d+6}{4} \rfloor \leq f(d) \leq \lceil \frac{d+1}{3} \rceil$.

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   - **(Alon, Ding, Oporowski, Vertigan; Haxell, Szabó, Tardos)** $\lfloor \frac{d+6}{4} \rfloor \leq f(d) \leq \lceil \frac{d+1}{3} \rceil$.

The *connected tree-depth* of a graph H is the minimum depth of a rooted tree T such that H is a subgraph of the closure of T.

Conjecture: (Norin, Scott, Seymour, Wood) If the connected tree-depth of H is at most t, then the class of H-minor free graphs is clustered $(2t-2)$-colorable.

THANK YOU!