On some properties of the α-spectral radius of (hyper)graph

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August 19, 2019

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Outline

1. Notation and preliminaries
2. Graph operations and α-spectral radius of hypergraph
3. Supertrees with Large α-Spectral Radius
4. Majorization and α- spectral radius of graph

Notation and preliminaries

1. A k-uniform hypergraph $H = (V(H), E(H))$ consists of the vertex set $V(H)$ and the edge set $E(H)$ which is a collection of k-subsets of $V(H)$.
2. For $v \in V(H)$, the degree of v,denoted by $d_H(v)$, is defined as the cardinality of $\{e \in E(H) | v \in e\}$.
3. Let $G = (V, E)$ be a simple graph. The kth power hypergraph [1] of G is the k-uniform hypergraph resulting from adding $k-2$ new vertices to each edge of G. The kth power of a tree is called a hypertree [1].
4. A supertree [2] is a hypergraph which is both connected and acyclic. A supertree is called a caterpillar if the removal of all pendant edges results in a loose (linear) path. Otherwise, it is called a non-caterpillar.

[1] Shenglong Hu, Liqun Qi, and Jia-Yu Shao. "Cored Hypergraphs, Power Hypergraphs and Their Laplacian H-Eigenvalues". In: *Linear Algebra and Its Applications* 439.10 (2013), pp. 2980–2998.

[2] Honghai Li, Jia-Yu Shao, and Liqun Qi. "The extremal spectral radii of k-uniform supertrees". In: *Journal of Combinatorial Optimization* 32.3 (2016), pp. 741–764.

Eigenvalues of Tensor

1. A tensor (hypermatrix) $\mathscr{A} = (a_\alpha)$ is a multi-array of entries $a_\alpha \in F$, where $\alpha \in [n]^m$ and $\mathbf{F}$ is a field.
2. In 2005, Qi [3] introduced the definition of eigenvalues of a tensor.

$$\mathcal{A}\boldsymbol{x} = \lambda \boldsymbol{x}^{[k-1]}. \tag{1}$$

   λ is called an eigenvalue of $\mathcal{A}$, $\boldsymbol{x} \in \mathbb{C}^n$ is called an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ, where $\mathcal{A}\boldsymbol{x} \in \mathbb{C}^n$ and
   $$(\mathcal{A}\boldsymbol{x})_i = \sum_{\beta \in [n]^{k-1}} a_{i,\beta} \prod_{j \in \beta} x_j, \quad \boldsymbol{x}^{[r-1]} = \left(x_1^{r-1}, x_2^{r-1}, \ldots, x_n^{r-1}\right)^{\mathrm{T}}.$$
3. The spectral radius of $\mathcal{A}$ is the largest modulus of its eigenvalues, denoted by $\rho(\mathcal{A})$.

[3] Liqun Qi. "Eigenvalues of a real supersymmetric tensor". In: *J. Symbolic Comput.* 40.6 (2005), pp. 1302–1324.

Tensors related with Hypergraphs

1. (J. Cooper,2012 [4]) The adjacency tensor $\mathscr{A}(G) = (a_{i_1 \ldots i_k})$ of a k-uniform hypergraph G is defined to be a k-th order n dimensional non-negative tensor with entries $a_{i_1 \ldots i_k}$ such that

$$a_{i_1 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

2. The degree tensor $\mathscr{D} = \mathscr{D}(G)$ of G, is a kth order n-dimensional diagonal tensor, with its ith diagonal element as $d(i)$.
3. The Laplacian tensor $\mathscr{L}$ of G is defined by $\mathscr{D} - \mathscr{A}$; The signless Laplacian tensor $\mathscr{Q}$ of G is defined by $\mathscr{D} + \mathscr{A}$.
4. Nonnegative tensors, Laplacian tensors are both copositive tensors(协正张量). Namely $\mathbf{x}^T \mathscr{A} \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n_+$.

[4] Joshua Cooper and Aaron Dutle. "Spectra of Uniform Hypergraphs". In: *Linear Algebra and its applications* 436.9 (2012), pp. 3268–3292.

α-matrix (tensor) for (hyper)graph

1. For an ordinary undirected graph G, 2017, Nikiforov [5] defined the matrix $A_\alpha(G)$ as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G) \text{ for any real } \alpha \in [0,1].$$

2. (2018, Guo,Zhou [6]) For $0 \le \alpha < 1$, the $\mathscr{A}_\alpha$ tensor of a k-uniform hypergraph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is defined as follows:

$$\mathscr{A}_\alpha(G) = \alpha\mathscr{D}(G) + (1-\alpha)\mathscr{A}(G). \quad (2)$$

- if $\alpha = 0$, then $\mathscr{A}_\alpha(G) = \mathscr{A}(G)$;
- if $\alpha = 1/2$, then $\mathscr{A}_\alpha(G) = \frac{1}{2}\mathscr{Q}(G)$;
- if $\alpha = 1$, then $\mathscr{A}_\alpha(G) = \mathscr{D}(G)$;
- $$\mathscr{A}_\alpha(G) - \mathscr{A}_\beta(G) = (\alpha-\beta)\mathscr{L}(G) \quad (3)$$

[5] V Nikiforov. "MERGING the A-and Q-SPECTRAL THEORIES". In: *Applicable Analysis and Discrete Mathematics* 11.1 (2017), pp. 81–107.

[6] HaiYan Guo and Bo Zhou. "On the α-Spectral Radius of Uniform Hypergraphs". In: *arXiv preprint arXiv:1807.08112* (2018).

Homogeneous polynomial form for $\mathscr{A}_\alpha$(G)

For a vector $\boldsymbol{x}$ of dimension n and subset $U \subseteq [n] = \{1, 2, \ldots, n\}$. Denote $x_U = \prod_{i \in U} x_i$. We have

$$\boldsymbol{x}^T \mathcal{A}_\alpha(G)\boldsymbol{x} = \sum_{e \in E(G)} (\alpha \sum_{u \in e} x_u^k + (1-\alpha)k x_e). \tag{4}$$

Let $\boldsymbol{x}$ be the unit Perron eigenvector of $\mathcal{A}_\alpha(G)$. We have:

$$\rho_\alpha(G) = \sum_{e \in E(G)} (\alpha \sum_{u \in e} x_u^k + (1-\alpha)k x_e) \tag{5}$$

$$\rho_\alpha(G) x_v^k = \alpha d_v x_v^k + (1-\alpha) \sum_{e \in E_G(v)} x_e. \tag{6}$$

Let G be ordinary undirected graph of order n. For the α eigenvalues of G:

$$\lambda_1 (A_\alpha(G)) \geq \lambda_2 (A_\alpha(G)) \geq \cdots \geq \lambda_n (A_\alpha(G))$$

Nikiforov [5] obtain the following result:

Proposition 1.1 (Monotonicity of $\lambda_k(A_\alpha(G))$ in α for graph G)

If $1 > \alpha > \beta > 0$ and G is a graph of order n, then

$$\lambda_k (A_\alpha(G)) \geq \lambda_k (A_\beta(G))$$

for any $k \in [n]$. If G is connected, then inequality is strict, unless $k = 1$ and G is regular.

From Formula (3) and Laplacian tensors is copositive, we have

Proposition 1.2 (Monotonicity of $\rho(\mathscr{A}_\alpha(G))$ in α for hypergraph G)

If $1 > \alpha > \beta > 0$ and G is a hypergraph of order n, then

$$\rho\left(A_\alpha(G)\right) \geq \rho\left(A_\beta(G)\right)$$

Proposition 1.3

If G is a connected hypergraph with maximal degree Δ and $\alpha \in [0,1]$, then

$$\rho(\mathscr{A}(G)) \leq \rho(\mathscr{A}_\alpha(G)) \leq \Delta.$$

Furthermore, if $\rho(\mathscr{A}_\alpha(G)) = \Delta$, then either $\alpha = 1$, or G is regular.

Outline

Graph operations and $\rho_\alpha(G)$

From Perron-Frobenius theorem for nonnegative tensors, we know that the α-spectral radius is monotone with respect to deletion of edges.

Lemma 2.1 (Edge deletion operation)

If G is a connected k-uniform hypergraph and $e \in E(G)$, then $\rho_\alpha(G) > \rho_\alpha(G-e)$.

Consequently, among k-uniform hypergraphs on n vertices the complete hypergraph $K_n^{(k)}$ has the maximum α-spectral radius.

Lemma 2.2 (The edge rotation lemma (2018, Guo, Zhou [6]))

For $k \geq 2$, let G be a k-uniform hypergraph with $u, v_1, \ldots, v_r \in V(G)$ and $e_1, \ldots, e_r \in E(G)$ for $r \geq 1$ such that $u \notin e_i$ and $v_i \in e_i$ for $i = 1, \ldots, r$, where $v_1, \ldots, v_r$ are not necessarily distinct.
Let $e'_i = (e_i \backslash \{v_i\}) \cup \{u\} \notin E(G)$ for $i = 1, \ldots, r$.
Let $G' = G - \{e_1, \ldots, e_r\} + \{e'_1, \ldots, e'_r\}$. If the α-Perron vector $\boldsymbol{x}$ of G satisfies $x_u \geq \max\{x_{v_1}, \ldots, x_{v_r}\}$, then $\rho_\alpha(G') > \rho_\alpha(G)$.

Corollary 2.1

Let u_1, u_2 are non-pendant vertices in an edge of connected uniform hypergraph H with $|E_H(u_i) \backslash E_H[\{u_1, u_2\}]| \geq 1$ for $i = 1, 2$. Let H' be the hypergraph obtained from H by moving edges $E_H(u_2) \backslash E_H[\{u_1, u_2\}]$ from u_2 to u_1 and $H \ncong H'$, then

$$\rho_\alpha(H) < \rho_\alpha(H').$$

Definition 2.3 (Edge-releasing Operation)

Let G be a k-uniform linear hypergraph with $k \geq 3$. Let $e = \{v_1, \ldots, v_k\}$ be an edge of G with $d_G(v_i) \geq 2$ for $i = 1, \ldots, r$, and $d_G(v_i) = 1$ for $i = r+1, \ldots, k$, where $3 \leq r \leq k$. Let G' be the hypergraph obtained from G by moving all edges containing $v_3, \ldots, v_r$ except e from $v_3, \ldots, v_r$ to v_1. We say G' is obtained from G by Operation I.

Theorem 2.4 (2018, Guo,Zhou)

If G' is obtained from G by Operation I, then we have $\rho_\alpha(G') > \rho_\alpha(G)$.

Definition 2.5 (2-switching operation)

Let e_1, e_2 be two edges of k-uniform hypergraph $G = (V, E)$.
If U_1, U_2 are r-subsets of e_1, e_2 respectively with $1 \leq r < k$ and k-sets $e_1' = (e_1 \cup U_2) \backslash U_1, e_2' = (e_2 \cup U_1) \backslash U_2 \notin E(G)$.
We say that hypergraph $G' = G - \{e_1, e_2\} + \{e_1', e_2'\}$ is obtained from G by 2-switching operation $e_1 \underset{U_2}{\overset{U_1}{\rightleftharpoons}} e_2$.

Theorem 2.6 (Wang,Shan, et al. 2019+; Guo,Zhou, 2018)

Let G be a connected k-uniform hypergraph and $e, f \in E(G)$. Let $U_1 \subset e, U_2 = e \backslash U_1, V_1 \subset f, V_2 = f \backslash V_1$ with $1 \leq |U_1| = |V_1| \leq k - 1$. Suppose that $e' = U_1 \cup V_2$ and $f' = V_1 \cup U_2$ are k-subsets of $V(G)$ and not in $E(G)$. Let $G' = G - \{e, f\} + \{e', f'\}$. Let $\boldsymbol{x}$ be the principal eigenvector of $\mathcal{A}_\alpha(G)$. If $x_{U_1} \geq x_{V_1}$ and $x_{U_2} \leq x_{V_2}$, then $\rho_\alpha(G) \leq \rho_\alpha(G')$. Moreover, the equality holds iff $x_{U_2} = x_{V_2}$ and $x_{U_1} = x_{V_1}$.

Corollary 2.2

Let G be a connected k-uniform hypergraph and $\boldsymbol{x}$ be the principal eigenvector of $\mathcal{A}_\alpha(G)$. Suppose that $e, f \in E(G)$ such that

$$\{u_1, u_2\} \subset e, \{v_1, v_2\} \subset f \text{ and } x_{u_1} > x_{v_1}, x_{u_2} \leq x_{v_2}.$$

If u_i is not adjacent to v_j in G for any $i, j \in \{1, 2\}$, then there exist k-subsets e', f' of $V(G)$ with $\{u_1, v_2\} \subset e', \{u_2, v_1\} \subset f'$ such that:

$$\rho_\alpha(G) < \rho_\alpha(G'),$$

where $G' = G - \{e, f\} + \{e', f'\}$.

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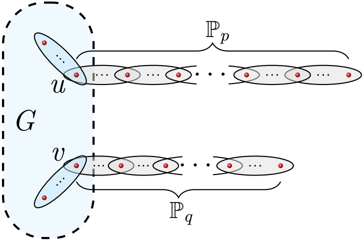


Figure 1: The hypergraph $G_{u,v}(p, q)$

For non-negative integers a, b, c, d with $a + b = c + d$, we say that the graph $G_{u,v}(c, d)$ is obtained from $G_{u,v}(a, b)$ by an edge grafting operation on the two relevant pendant paths of $G_{u,v}(a, b)$.
The edge grafting operation for graphs was usually considered in the study on graph variants.

Nikiforov's Conjecture on grafting operation

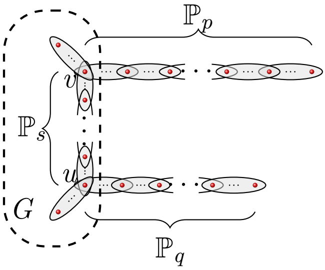


Figure 2: Hypergraph $G_{u,v}(p, q; s)$

Conjecture 1 (Nikiforov and Rojo,2018 [5])

Suppose G be an ordinary graph with $u, v \in V(G)$ and $d_G(u), d_G(v) \geq 2$. Let $\alpha \in [0, 1)$ and $s = 0, 1$. If $q \geq 1$ and $p \geq q + 2$, then

$$\rho_\alpha(G_{u,v}(p, q; s)) < \rho_\alpha(G_{u,v}(p - 1, q + 1; s)).$$

Question 1

For which connected graphs G the following statement is true:
Let $\alpha \in [0,1)$ *and let* u *and* v *be non-adjacent vertices of* G *of degree at least* 2*. If* $q \geq 1$ *and* $p - q \geq 2$*, then*
$\rho_\alpha(G_{u,v}(p,q)) < \rho_\alpha(G_{u,v}(p-1,q+1))$.

Conjecture 1 is already confirmed by Lin in [7]. Furthermore, Guo in [5] showed that Conjecture 1 also holds for connected k-uniform hypergraph when $s = 0$.
Question 1 is confirmed true by Lin in [9] for the type of graphs $G_{u,v}(p,q;s)$ when $p - q \geq \max\{s+1, 2\}$.

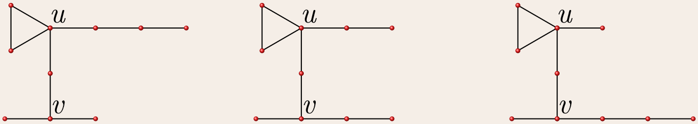
[7] Huiqiu Lin, Xing Huang, and Jie Xue. "A Note on the A_α-Spectral Radius of Graphs". In: *Linear Algebra and its Applications* 557 (2018), pp. 430–437.

Theorem 2.7 (Wang,Shan, et al. 2019+)

Let u, v be two non-pendant vertices of hypergraph G. If there exist an internal path $\mathbb{P}$ with s length in hypergraph $G_{u,v}(p,q)$ for any $p \geq q \geq 1$, then we have

$$\rho_\alpha(G_{u,v}(p+1,q-1)) < \rho_\alpha(G_{u,v}(p,q)) \qquad \text{for } p-q+1 \geq s \geq 0.$$

Remark 1



$G_{u,v}(p+1,q-1)$ $\quad$ $G_{u,v}(p,q)$ $\quad$ $G_{u,v}(p-1,q+1)$

Take $\alpha = 1/2, p = q = 2, s = 2.$ $p - q + 1 = 1 < s.$
By direct calculation (using Sagemath), we have
$\rho_\alpha(G_{u,v}(3,1)) \doteq 2.7325,\ \rho_\alpha(G_{u,v}(2,2)) \doteq 2.7307,\ \rho_\alpha(G_{u,v}(1,3)) \doteq 2.7033.$
$\rho_\alpha(G_{u,v}(p+1,q-1)) > \rho_\alpha(G_{u,v}(p,q)) > \rho_\alpha(G_{u,v}(p-1,q+1)).$

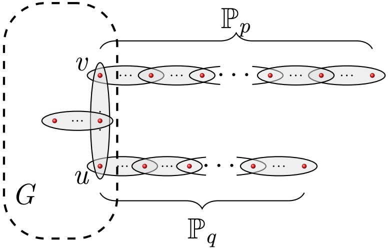


Figure 3: The hypergraph $G_{u,v}(p, q)$

Theorem 2.8

Let G be a connected uniform hypergraph and u, v be two pendant vertices in a pendant edge e of G. If $p \geq q \geq 1$, then

$$\rho_\alpha(G_{u,v}(p, q)) > \rho_\alpha(G_{u,v}(p + 1, q - 1)).$$

Outline

Supertrees with Large α-Spectral Radius

1. In [8], [9],Yuan et al. determined the top ten supertrees with the maximum spectral radii.
2. Guo, Zhou, 2018, Suppose that $k \geq 2$. If G is a k-uniform supertree with $m \geq 1$ edges, then $\rho_\alpha(G) \leq \rho_\alpha(K_{1,m}^k)$ for $0 \leq \alpha < 1$ with equality holds if and only if $G \cong K_{1,m}^k$.

Theorem 3.1 (Wang, Shan, et al. 2019+)

Let T be a k-uniform supertree on n vertices with m edges, suppose that $T \ncong K_{1,m}^k$, then we have

$$\rho_\alpha(T) \leq \rho_\alpha(S^k(1, m-2))$$

where the equality holds if and only if $T \cong S^k(1, m-2)$.

[8] Xiying Yuan, Jiayu Shao, and Haiying Shan. "Ordering of some uniform supertrees with larger spectral radii". In: *Linear Algebra Appl.* 495 (2016), pp. 206–222.

[9] Xiying Yuan, Xuelian Si, and Li Zhang. "Ordering Uniform Supertrees by Their Spectral Radii". In: *Frontiers of Mathematics in China* 12.6 (2017), pp. 1393–1408.

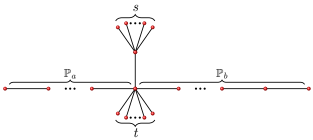


Figure 4: The tree $T(s, t; a, b)$

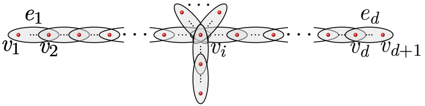


Figure 5: $H_1(m, d) = T^k(1, m-d-2; \left\lfloor \frac{d}{2} \right\rfloor, \left\lceil \frac{d}{2} \right\rceil)$

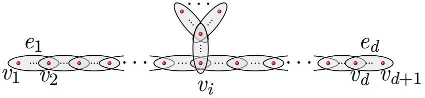


Figure 6: $H_2(m, d) = T^k(m-d-1, 0; \left\lfloor \frac{d}{2} \right\rfloor, \left\lceil \frac{d}{2} \right\rceil)$

Let $NC(m)$ denote the set of k-uniform non-hyper-caterpillars with m edges. Let $C(m,d)$ and $NC(m,d)$ denote, respectively, the set of k-uniform hyper-caterpillars and non-hyper-caterpillars with m edges and diameter d. In [10], Guo and Zhou investigated the adjacency spectral radius of uniform hypertrees and showed that $H_1(m,d)$ be the unique non-hyper-caterpillar with maximum spectral radius among $NC(m,d)$.

Theorem 3.2

Let $G \in NC(m,d)$, then we have

$$\rho_\alpha(G) \leq \max\{\rho_\alpha(H_1(m,d)), \rho_\alpha(H_2(m,d))\}.$$

Equality holds if and only if $G \cong H_1(m,d)$ or $G \cong H_2(m,d)$.

[10] Haiyan Guo and Bo Zhou. "On the spectral radius of uniform hypertrees". In: *Linear Algebra and its Applications* 558 (2018), pp. 236–249.

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Majorization and α- spectral radius of graph

Definition 4.1

For non-increasing sequences $\pi = (n_1, \ldots, n_k)$ and $\pi' = \left(n'_1, \ldots, n'_k\right)$, π' is said to majorize π if $\sum_{i=1}^{k} n_i = \sum_{i=1}^{k} n'_i$, and

$$\sum_{i=1}^{p} n_i \geq \sum_{i=0}^{p} n'_i \text{ for } p = 1, \ldots, k-1$$

π is said to majorize π' and denoted by $\pi \rhd \pi'$ or $\pi' \lhd \pi$.

For details, the readers are referred to the book of Marshall and Olkin [11]

Proposition 4.1

Let $\pi = (n_1, n_2 \cdots n_k)$ *and* $\pi' = \left(n'_1, \cdots, n'_k\right)$ *be two different non-increasing integer sequences. If* $\pi' \lhd \pi$, *then there exists a series integer sequences* $\pi_1, \pi_2, \ldots, \pi_k$ *such that* $\pi' \lhd \pi_1 \lhd \cdots \lhd \pi_k \lhd \pi$, *and only two components of* π_i *and* π_{i+1} *are different from 1.*

[11] Barry C. Arnold (auth.) Albert W. Marshall Ingram Olkin. *Inequalities: Theory of majorization and its applications*. 2nd ed. Springer Series in Statistics. Springer-Verlag New York, 2011.

Let $G_v(n_1, n_2, \ldots, n_k)$ be the graph obtained from a non-trivial connected graph G by attaching pendant paths of lengths $n_1, n_2, \ldots, n_k$,respectively, at some vertex $v \in V(G)$.

From Theorem 2.7, we have the following corollary which reported in [6]

Corollary 4.1 (Guo,Zhou [6])

For $k \geq 2$, let G be a connected k-uniform hypergraph with $|E(G)| \geq 1$ and $u \in V(G)$. For $p \geq q \geq 1$ and $0 \leq \alpha < 1$, we have $\rho_\alpha(G_u(p, q)) > \rho_\alpha(G_u(p+1, q-1))$.

According to Proposition 4.1 and Corollary 4.1, we have:

Theorem 4.2

Suppose $\pi = (n_1, n_2 \cdots n_k)$ and $\pi' = \left(n'_1, \cdots, n'_k\right)$ be two different non-increasing integer sequences such that $\pi' \lhd \pi$. Then

$$\rho_\alpha(G_v(n_1, n_2, \ldots, n_k) < \rho_\alpha(G_v(n'_1, \cdots, n'_k) \text{ for } 0 \leq \alpha < 1.$$

The main result of [12] is an immediate consequence of the theorem above.

[12] Mohammad Reza Oboudi. "Majorization and the spectral radius of starlike trees". In: *Journal of Combinatorial Optimization* 36.1 (2018), pp. 121–129.

Let $G = K_{n_1,\ldots,n_k}$ be complete k-partite graph of n vertices which fall into k disjoint blocks V_i of size n_i, respectively. Then $\pi = (V_1, V_2, \ldots, V_k)$ is an equitable partition of $A_\alpha(G)$.

In [13], [14], the relationship between the spectral radius and the energy of complete multipartite graphs are studied.

1. (Dragan Stevanović, Ivan Gutman,[15]) If $n_i - n_j \geq 2$, then $\rho(A(K_{n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_p})) > \rho(A(K_{n_1,\ldots,n_i,\ldots,n_j,\ldots,n_p}))$
2. For fixed n, both the spectral radius and the energy of complete p-partite graphs are minimal for complete split graph $CS(n, p-1)$ and are maximal for Turán graph $T(n, p)$.
3. (Mohammad Reza Oboudi [16]) If $n_i - n_j \geq 2$, then $\rho(Q(K_{n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_p})) > \rho(Q(K_{n_1,\ldots,n_i,\ldots,n_j,\ldots,n_p}))$

[13] Dragan Stevanović, Ivan Gutman, and Masood U. Rehman. "On spectral radius and energy of complete multipartite graphs". In: *Ars Math. Contemp.* 9.1 (2015), pp. 109–113.

[14] Mohammad Reza Oboudi. "A relation between the signless Laplacian spectral radius of complete multipartite graphs and majorization". In: *Linear Algebra and its Applications* 565 (2019), pp. 225–238.

Theorem 4.3

Let π be an equitable partition of square matrix A and $B = (b_{ij})_{s\times s}$ be the quotient matrix A/π, Then

1. $\phi(A, x) = \phi(B, x)\prod_{i=1}^{m}(x - b_{ii})^{-1}\phi(A_{ii}, x)$.
2. $\rho(A) = \rho(B) = \max_{\|X\|=1} X^T Q_s(A)X$ *if A is nonnegative matrix, where $Q_s(A)$ is the symmetric quotient matrix of A corresponding to partition π.*

Corollary 4.2

Let $k \geq 2$ and $n_1, \ldots, n_k$ be some positive integers. Let $n = n_1 + \cdots + n_k$. Then

1. $\phi(B, x) = f(x)\prod_{i=1}^{k}(x + n_i - n\alpha)$, *where* $f(x) = 1 - \sum_{i=1}^{k}\frac{(1-\alpha)n_i}{x+n_i-n\alpha}$
2. $\phi_\alpha(K_{n_1,\ldots,n_k}, x) = \prod_{i=1}^{k}(x + n_i\alpha - n\alpha)^{n_i-1}\phi(B, x)$

Theorem 4.4 (Shan,Shi, 2019+)

Let $k \geq 3$ and $n'_1, \ldots, n'_k$ and $n_1, \ldots, n_k$ be some positive integers such that $n'_1 \geq \cdots \geq n'_k$ and $n_1 \geq \cdots \geq n_k$. If $(n'_1, \ldots, n'_k) \lhd (n_1, \ldots, n_k)$, then

1. *when $0 \leq \alpha < 1 - \frac{1}{k}$, $\rho_\alpha(K_{n_1,\ldots,n_k}) > \rho_\alpha(K_{n'_1,\ldots,n'_k}) > n\alpha$;*
2. *when $\alpha = 1 - \frac{1}{k}$, $\rho_\alpha(K_{n_1,\ldots,n_k}) = \rho_\alpha(K_{n'_1,\ldots,n'_k}) = n\alpha$;*
3. *when $0 \leq \alpha > 1 - \frac{1}{k}$, $\rho_\alpha(K_{n_1,\ldots,n_k}) < \rho_\alpha(K_{n'_1,\ldots,n'_k}) < n\alpha$*

Thank you!