(k, d)-choosable of graphs

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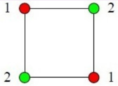
vertex coloring

Definition

A *proper k-coloring* of G is a mapping $\phi : V(G) \to \{1, 2, \cdots, k\}$ such that $\phi(x) \neq \phi(y)$ for every $xy \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the least number of colors in an proper vertex coloring of G.



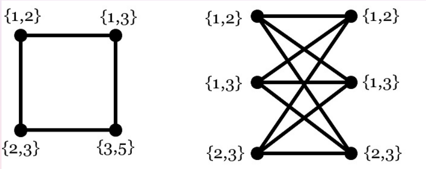
It is well-known that if G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$. Every planar graph is 4-vertex-colorable(四色定理).

一个平面图G是3-点可染的，如果满足下列条件之一：

- G不含5-圈，同时两个3-圈的距离至少为2;
- G不含5-, 7-圈，同时两个三角形不相邻[J C T(B) 96 (2006) 958 – 963];

列表染色的定义

A graph G is said to be f-choosable if, whenever we give lists of $f(x)$ colors to each vertex $v \in V(G)$, there exists a proper vertex coloring of G where each vertex is colored with a color from its own list. If $f(x) = k$ for every vertex $x \in V(G)$, we say that G is k-choosable. The *choice number* or the list chromatic number $\chi_{list}(G)$ is the smallest integer k such that G is k-choosable.



The concept of a list coloring was introduced by Vizing[Metody Diskret. Analiz, Novosibirsk 29 (1976) 3-10] and Erdős, Rubin and Taylor[Congr. Numer., 26 (1979), 125 – 157], respectively. It is obvious that $\chi_{list}(G) \leq \Delta + 1$.

Every planar graph is 5-choosable

C. Thomassen, Every planar graph is 5-choosable, J Comb Theory (B) 62 (1994) 180-181.

If G is a plane graph with outer cycle C, p_1 and p_2 are two adjacent vertices on C, and L is a list assignment for G such that $|L(v)| \geq \backslash 5$ for all $v \in V(G) \backslash V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \backslash V(P)$, $|L(p_1)| = |L(p_2)| = 1$ and $L(p_1) \neq L(p_2)$, then G is L-colorable.

It implies that $\chi_{list}(G) \leq 5$ for any planar graph G. The result is best possible, see[M. Voigt, Disc Math 120 (1993) 215-219].

L. Postle and R. Thomas, J. Comb. Theory, Ser. B 111(2015)234 - 241 .

If G is a plane graph with outer cycle C, $v_1, v_2 \in V(C)$ and L is a list assignment for G with $|L(v)| \geq 5$ for all $v \in V(G) \backslash V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \backslash \{v_1, v_2\}$, and $|L(v_1)| = |L(v_2)| = 2$, then G is L-colorable.

Z Dvořák, *etc*, 5-list-coloring planar graphs with distant precolored vertices, J Comb Theory, B 122(2017)311 - 352.

If G is a planar graph with list assignment L that gives lists of size one or five to its vertices and the distance between any pair of vertices with lists of size one is at least 20780, then G is L-colorable.

局部平面图的5列表染色

Z Dvořák, *etc*, 5-choosability of graphs with crossings far apart, J Comb Theory, B 123(2017)54－96.

Every graph drawn in the plane so that the distance between every pair of crossings is at least 15 is 5-choosable.

设图G已经嵌入到曲面S中，G的边宽(edge-width)是S中不可收缩的最短圈的长度，局部平面图G是指它嵌入到曲面S后使得它的边宽相对曲面的亏格来说要大很多，也就是说一些长度相对小的圈的导出子图在曲面S上一定是平面图. 1993年Thomassen证明了：局部平面图是5-可染的；2008年Mohar等人推广地证明了下面的结果：

M DeVos, K Kawarabayashi, B Mohar, Locally planar graphs are 5-choosable, J Comb. Theory, Series B 98 (2008) 1215－1232.

Every graph embedded in a fixed surface with sufficiently large edge-width is 5-choosable.

不含K_5作为子式的图是5列表可染的

Let $e = xy$ be an edge of a graph $G = (V, E)$. To *contract* an edge e of a graph G is to delete the edge and then (if the edge is a link) identify its ends. A graph H is a *minor* of a graph G if G has a subgraph contractible to H; G is called *H-minor free* if G does not have H as a minor.

R Škrekovski, Choosability of K_5-minor-free graphs, Disc Math 190 (1998) 223-226.
W J He, W J Miao, Y F Shen, Another proof of the 5-choosability of K_5-minor-free graphs, Disc Math 308 (2008) 4024 – 4026.

Let G be a K_5-minor-free graph and let L be a list assignment of G such that $|L(v)| \geq 5$ for every vertex $v \in V(G)$. Suppose that H is a subgraph of G isomorphic to K_2 or K_3 and suppose that λ is an L-coloring of H. Then λ can be extended to an L-coloring of G.

符号平面图是5列表可染的

Let G be a graph and $\sigma : E(G) \to \{1, -1\}$ be a mapping. The pair (G, σ) is called a signed graph, and σ is called a signature of G. An edge e is *positive* (or *negative*) if $\sigma(e) = 1$ (or $\sigma(e) = -1$). Proper coloring means that for any edge $uv \in E(G)$, $f(u) \neq \sigma(uv) f(v)$.

L G Jin, Y L Kang, E Steffen, Choosability in signed planar graphs, Europ J Comb 52 (2016) 234 – 243

Every signed planar graph is 5-choosable and that there is a signed planar graph which is not 4-choosable while the unsigned graph is 4-choosable. For each $k \in \{3, 4, 5, 6\}$, every signed planar graph without circuits of length k is 4-choosable. Furthermore, every signed planar graph without circuits of length 3 and of length 4 is 3-choosable. They construct a signed planar graph with girth 4 which is not 3-choosable but the unsigned graph is 3-choosable.

有关3-可选的部分结果

N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992), 125 – 134.

If G is a bipartite planar graph, then $\chi_{list}(G) \leq 3$.

C. Thomassen, 3-list-coloring planar graphs of girth 5, J. Combin. Theory Ser. B 64(1995) 101-107.
A short list color proof of Grötzsch’s theorem, J. Combin. Theory Ser. B 88 (2003) 189 – 192

If G is a planar graph of girth $g \geq 5$, then $\chi_{list}(G) \leq 3$.

这个结果对围长是最好可能的，见[M. Voigt, A not 3-choosable planar graph without 3-cycles, Discrete Math. 146 (1995) 325-328].

李相文, On 3-choosable planar graphs of girth at least 4, Discrete Mathematics 309 (2009) 2424-2431

If a planar graph G has neither intersecting 4-cycles nor a 5-cycle intersecting with any 4-cycle, then G is 3-choosable.

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有关3-可选的部分结果

Let G be a planar graph. Then G is 3-choosable if one of the following conditions holds.

- G contains no cycles of length in $\{4,5,7,9\}$; [L. Zhang, B. Wu, Graph Theory Notes of New York 46 (2004) 27 – 30]
- G contains no cycles of length in $\{4,5,6,9\}$; [L. Zhang, B. Wu, Discrete Mathematics 297 (2005) 206 – 209]
- G contains no cycles of length in $\{4,6,8,9\}$; [L Shen, Y Q Wang, Information Processing Letters 104 (2007) 146 – 151]
- G contains no cycles of length in $\{4,6,7,9\}$; [Y Q Wang, H J Lu, M Chen, Information Processing Letters 105 (2008) 206 – 211]
- G contains no cycles of length in $\{4,5,8,9\}$; [Y Q Wang, H J Lu, M Chen, Disc Math, 310(2010),147-158]
- G contains no cycles of length in $\{4,7,8,9\}$. [Y Q Wang, Q Wu, L Shen, Disc Appl Math, 159(2011), 232-239]

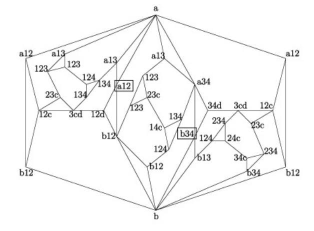
有关3-可选的部分结果

Let G be a planar graph. Then G is 3-choosable if one of the following conditions holds.

- G contains no cycles of length $3, 5$ and 6; [P Lam, W C Shiu, Z M Song, Discrete Math. 294 (2005) 297 – 301.]
- G contains no cycles of length 4 and 5 and every two triangles has distance at least 4, or G contains no cycles of length $4, 5$ and 6 and any two triangles has distance at least 3; [王维凡等, Disc Math 306(2006) 573 – 579]
- G contains no cycle of length at most 10 with a chord; [W F Wang, Taiwanese J Math, 11(2007) 179-186]
- G contains no cycles of length $5, 6$ and 7 and any two triangles has distance at least 3, or G contains no cycles of length $5, 6$ and 8 and any two triangles has distance at least 2; [H H Zhang Z R Sun, Information Processing Letters 107 (2008) 102 – 106]
- G contains no cycles of length $3, 7$ and 8. [Z Dvoraka, B Lidicky, R Skrekovski, Discrete Mathematics 309 (2009) 5899-5904]

同时王维凡等在他们的论文中指出：there exists a non-3-choosable planar graph without 4-cycles, 5-cycles and intersecting triangles. 在文[D Q Wang, Y P Wen, K L Wang, Information Processing Letters 108 (2008) 87 – 89] 中给出了一个更小的例子.

Voigt[Discrete Mathematics 307 (2007) 1013 – 1015]首先给出了没有4- 圈和5- 圈的平面图是不能3- 可选的一个例子.



Z. Dvořák Journal of Combinatorial Theory,Series B 104(2014) 28 – 59

If G is a planar graph such that the distance between any two (≤ 4)-cycles is at least 26, then G is 3-choosable.

Z. Dvořák and L. Postle, J Comb Theory, B 129(2018)38 – 54

Every planar graph G without cycles of lengths 4 to 8 is 3-choosable.

P.C.B. Lam, B. G. Xu, J.Z. Liu, The 4-choosability of plane graphs without 4-cycles, J. Combin. Theory, Ser. B 76 (1999), 117-126.

If G is a planar graph without 4-cycles, then $\chi_{list}(G) \leq 4$.

P. C. B. Lam, W. C. Shiu, B. G. Xu, On structure of some plane graphs with applications to choosability, J. Combin. Theory Ser. B 82 (2001) 285-296
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If G is a planar graph without 5-cycles, then $\chi_{list}(G) \leq 4$.

G. Fijavz, M. Juvan, B. Mohar, R. Skrekovski, Planar graphs without cycles of specific lengths, European J. Combin. 23 (2002) 377 - 388.

If G is a planar graph without 6-cycles, then G is 3-degenerate and it follows that $\chi_{list}(G) \leq 4$.

有关4-可选的已知结果

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If G is a planar graph without intersecting 3-cycles(that is, every vertex is incident with at most one 3-cycle), then $\chi_{list}(G) \leq 4$.

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All planar graphs without triangular 4-cycles are 4-choosable.

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最近我们得到的三个结果

R Y Xu and W, A sufficient condition for a planar graph to be 4-choosable, Discrete Applied Mathematics 224 (2017) 120 – 122.

Let G be a planar graph. If every 5-cycle of G is not adjacent simultaneously to 3-cycles and 4-cycles, then G is 4-choosable.

D Q Hu and W, Planar graphs without intersecting 5-cycles are 4-choosable, Discrete Mathematics 340 (2017) 1788 – 1792

Planar graphs without intersecting 5-cycles are 4-choosable.

D Q Hu, D J Huang, W F Wang and W, A note on the choosability of planar graphs without chordal 6-cycles, Discrete Applied Mathematics 244 (2018) 116 – 123

Planar graphs without chordal 6-cycles are 4-choosable.

s-separated k-choosable((k,s)-列表染色)

A graph G is said to be (k,s)-*choosable* $(k \geq s)$ if for each list assignment L satisfying $|L(v)| \geq k$ for each vertex v and $|L(x) \cap L(y)| \leq s$ for each edge xy, G has an L-coloring(即在列表染色中增加了列表的要求：任意相邻两点的列表至多有s 种颜色相同). Let $\chi_l(G,s)$ denote the minimum k such that G is L-colorable for each s-separated k-list L.

Kratochvíl等J. Graph Theory, 27 (1998), 43-49.

(1) For positive integers s, n with $s \leq n$,

$$\sqrt{\frac{1}{2}sn} \leq \chi_l(K_n, s) \leq \sqrt{2esn}.$$

(2) $\chi_l(G,s) \leq \sqrt{2es(\Delta(G)-1)}$.
(3) Every planar graph is $(4,1)$-choosable([Disc Math 338 (2015) 1779 – 1783]里给出了一个更强的结果).

(4) Every triangle-free planar graph is $(3,1)$-choosable.

s-separated k-choosable((k,s)-可选的)

Thomassen proved that planar graphs are 5-choosable and hence they are $(5,d)$-choosable for all d. Voigt constructed a non-4-choosable planar graph and there are also examples of non-$(4,3)$-choosable planar graphs. Škrekovski observed that there are examples of triangle-free planar graphs that are not $(3,2)$-choosable.

Conjecture 1

Every planar graph is $(4,2)$-choosable.

It is proved for all planar graphs without chorded l-cycles, for each $l \in \{5,6,7\}$[Graphs and Combinatorics (2017) 33:751 – 787].

Conjecture 2

Every planar graph is $(3,1)$-choosable

It is proved for all planar graphs without 4-cycles adjacent to 4^--cycles[Bull. Malays. Math. Sci. Soc. (2018) 41:1507 – 1518], without 5- and 6-cycles[J. Graph Theory 81(3), 283 – 306 (2016)], or neither 6-cycles nor adjacent 4- and 5-cycles[J Comb Optim (2017) 34:987 – 1011].

Yue Wang, Jianliang Wu, Donglei Yang, Discrete Mathematics 342 (2019) 1782 – 1791

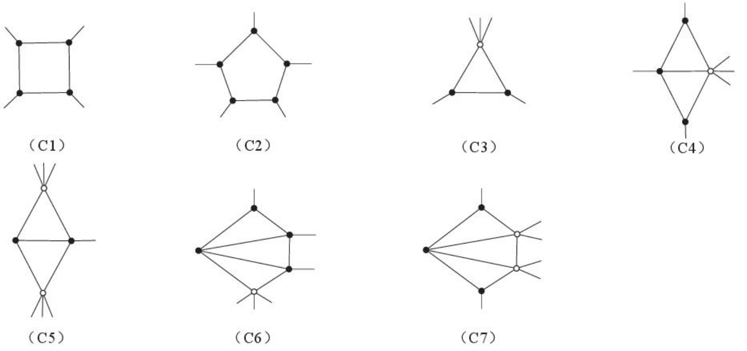
Every planar graph G is $(3,1)$-choosable if any i-cycle is not adjacent to a j-cycle, where $5 \leq i \leq 6$ and $5 \leq j \leq 7$.

证明思路

Let $G = (V, E, F)$ be a counterexample to the result with the fewest vertices. Then

(a) G has a 2^--vertex, or

(b) G contains one of the configurations $(C1)$-$(C19)$.



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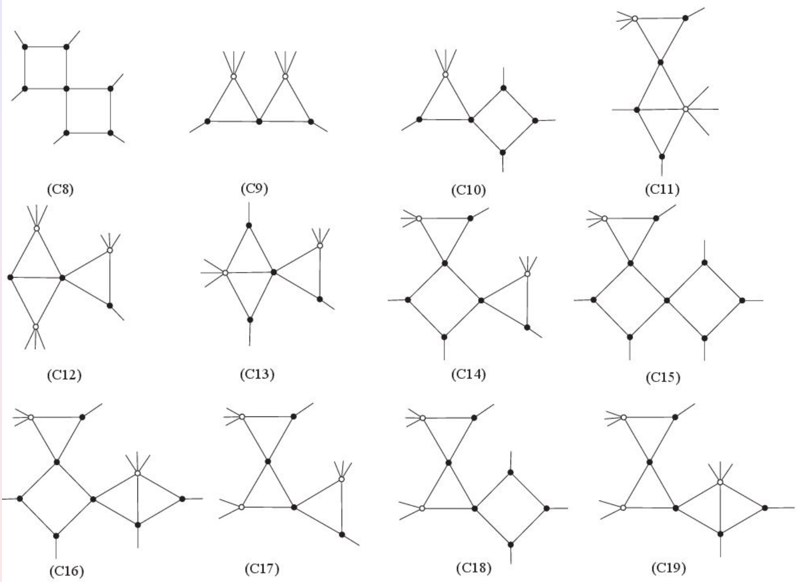
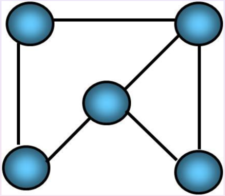


证明思路

Euler公式

$|V|-|E|+|F|=2$



$$|V|=5,\ |E|=6,\ |F|=3,\ |V|-|E|+|F|=5-6+3=2.$$

根据$\sum_{v\in V(G)} d(v)=2|E|$, $\sum_{f\in F(G)} d(f)=2|E|$, 我们得到

$$\sum_{v\in V}(d(v)-6)+\sum_{f\in F}(2d(f)-6)=-6(|V|-|E|+|F|)=-12<0. \tag{1}$$

$$\sum_{v\in V}(d(v)-4)+\sum_{f\in F}(d(f)-4)=-4(|V|-|E|+|F|)=-8<0. \tag{2}$$

By Euler's formula, we have the following formula.

$$\sum_{v\in V(G)}(2d(v)-6)+\sum_{f\in F(G)}(d(f)-6)=6(|E|-|V|-|F|)=-12<0.$$

Now we assign an initial charge $\mu(z)$ to each $z\in V(G)\cup F(G)$ by letting $\mu(v)=2d(v)-6$ for $v\in V(G)$, $\mu(f)=d(f)-6$ for $f\in F(G)$. Thus we have $\sum_{z\in V(G)\cup F(G)}\mu(z)<0$.

证明思路

We define the following two rounds of discharging rules. The first round contains (R1)-(R4), which are called *vertex rules*.

R1. Suppose $d(v) = 4$.

(R1A) If v is incident with exactly one 5^--face f, then v sends 2 to f.

(R1B) Suppose that $f_{5^-}(v) = 2$. If v is incident with a 3-face f_1 and a 4-face f_2 such that f_2 is incident with at least three 4^+-vertices, then v sends $\frac{4}{3}$ to f_1 and $\frac{2}{3}$ to f_2. Otherwise, v sends 1 to each incident 5^--face.

(R1C) If $f_{5^-}(v) = 3$, then v sends $\frac{2}{3}$ to each incident 5^--face.

R2. Suppose $d(v) = 5$.

(R2A) If $f_{5^-}(v) \le 2$, then v sends 2 to each incident 5^--face.

(R2B) If $f_{5^-}(v) = 3$ and $f_3(v) \le 1$, then v sends $\frac{3}{2}$ to its incident 3-face (if exists) and $\frac{5}{4}$ to each incident i-face, where $i \in \{4, 5\}$.

(R2C) If $f_{5^-}(v) = 3$ and $f_3(v) = 2$, then v sends $\frac{3}{2}$ to each incident 3-face and 1 to its incident i-face, where $i \in \{4, 5\}$.

(R2D) Suppose $f_3(v) = 3$. Assume that the faces incident with v are $f_1, f_2, \cdots, f_5$ in clockwise order such that f_1, f_2 are 3-faces. If f_3 or f_5 is a 3-face, then v sends $\frac{4}{3}$ to each incident 3-face. Otherwise f_4 is a 3-face. If f_4 is bad and at most one of f_1, f_2 is bad, then v sends $\frac{3}{2}$ to f_4 and $\frac{5}{4}$ to f_i $(i = 1, 2)$. Otherwise v sends 1 to f_4 and $\frac{3}{2}$ to f_i $(i = 1, 2)$.

R3. Suppose that $d(v) = 6, 7, 8$. If $f_{5^-}(v) < \lfloor \frac{3d(v)}{4} \rfloor$, then v sends 2 to each incident 5^--face. Otherwise v sends $\frac{3}{2}$ to each incident 5^--face.
R4. If $d(v) \geq 9$, then v sends 2 to each incident 5^--face.

The second round contains (R5), which is called the *face rule*.

R5. Let f be a d-face where $d \geq 7$. Let f_0, f_1, $f_2, \cdots$, f_{d-1} be the faces adjacent to f in clockwise order, and let $v_0, v_1, \cdots, v_{d-1}$ be the vertices incident with f in clockwise order such that v_i is incident with f_i and f_{i+1}, where the subscripts are taken modulo d here.

(R5A) f sends $\frac{d(f)-6}{d(f)}$ to f_i for any i $(0 \leq i < d)$.

(R5B) Suppose that f_i is hungry for some $i (0 \leq i < d)$. (1) If f_{i+1} is not hungry and $d(v_i) \leq 4$, then f_{i+1} sends $\frac{d(f)-6}{d(f)}$ to f_i. (2) If f_{i-1} is not hungry, $d(v_{i-1}) \leq 4$, and either $d(v_{i-2}) \geq 5$ or f_{i-2} is not hungry, then f_{i-1} also sends $\frac{d(f)-6}{d(f)}$ to f_i.

(k,s)-可选的

如果给图G的所有点v都分配一个颜色集合$L(v)$(也叫色列表)且G存在一个正常的点染色ϕ 使得对每个点$v \in V(G)$ 都有$\phi(v) \in L(v)$,则称图G是L-可染的. 如果对任何的色分配L, 满足$|L(v)| \geq k(\forall v \in V(G))$和$|L(x) \cup L(y)| \geq s(\forall xy \in E(G))$, G是L-可染的，则称G是(k,s)- 可选的.

每个平面图是(3, 11)- 可选的和(4,7)-可选的[Discrete Mathematics 341 (2018) 600 – 605].

在正常染色的基础上加上某个条件的染色

(1) 均匀染色(equitable coloring):任何两个不同的颜色所染的顶点数至多差1;
(2) 无圈点染色(acyclic coloring): 任何两个不同的颜色所染的点集合所导出的子图是一个森林;
(3) 线性染色(linear coloring): 任何两个不同的颜色所染的点集合所导出的子图是一个线性森林;
(4) (p,q)-标号((p,q)-labelling): 相邻的顶点的颜色至少差p, 距离为2的两个点的颜色至少差q;
(5) 邻点可区别的点染色(adjacent vertex distinguishing vertex coloring):任何相邻的两个点所对应的邻域的染色集合不同;
(6) 邻和可区别的点染色(adjacent sum distinguishing vertex coloring): 任何相邻的两个点所对应的邻域的颜色之和不相等;
(7) r-色调染色(r-hued coloring):度数为d的顶点邻域至少出现$min\{d, r\}$种颜色;
(8) 邻域r-限制染色(Neighborhood r-bounded coloring): 每个点的同色邻点数不得超过r个

两个分数方面的染色

- 圆染色(circular coloring): 用颜色$1, 2, ..., k$给图的每个点染色f. 如果对任意的$uv \in E(G)$,
  有$d \leq |f(u) - f(v)| \leq k - d (k \geq 2d \geq 1)$，则称$G$存在$(k, d)$-圆染色. 最小的$\frac{k}{d}$称为$G$ 的圆色数$\chi^C(G)$.
  等价定义：给一个周长为k的圆环$L(k \geq 2)$, 图G的每个点对应于L上长度为1的开弧. 如果两个点相邻, 它们对应的弧不交, 我们就说G是k-圆可染的. 最小的k称为G的圆色数.
  For any graph G, $\chi(G) - 1 < \chi^C(G) \leq \chi(G)$.
- 分数染色(fractional coloring): 用k种颜色给图的每个点染d种颜色. 如果任何相邻的两个点所染颜色的集合不交，则称G存在(k, d)-分数染色. 最小的k/d称为G的分数色数.

最近有个结果引起了不少的轰动：every planar graph has a $(9, 2)$-fractional coloring.

不要求正常情况下的染色

- $(k,d)^*$-染色: 用k种颜色去染图的点使得染同色的点集合的导出子图的最大度至多为d;
- 点荫度(vertex arboricity): 用k种颜色去染图的点使得染同色的点集合的导出子图是一个森林，所用最少的颜色数称为图的点荫度$va(G)$;
- 点线性荫度等(linear vertex arboricity): 用k种颜色去染图的点使得染同色的点集合的导出子图是一个线性森林，所用最少的颜色数k称为图的点线性荫度$vla(G)$;
- 圆点荫度(circular vertex arboricity): 用$k(\geq 2d)$种颜色去染图的点使得对每个$j(0 \leq j \leq k-1)$, 染$j, j+1, ..., j+d-1$的所有点的导出子图是一个森林(下标按k取模运算), 最小的k/d成为G的圆点荫度;

组合情况下的新的染色

- 均匀点荫度: 同色点集合的导出子图是一个森林，而且任何两个颜色所染的点数至多差1;
- 邻和可区别的(p,q)-标号: 给图G一个正常的点染色使得任何相邻的两个点所对应的邻域的颜色之和至少差p, 任何距离为2的两个点所对应的邻域的颜色之和至少差q;
- 邻和可区别的点荫度: 相同的邻和所导出的子图为森林;
- ...

还有game coloring, cochromatic number, achromatic number, antimagic label等.

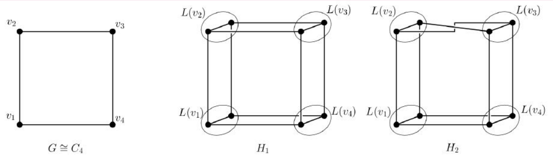
DP-coloring

Z. Dvořák and L. Postle, J Comb Theory, B 129(2018)38 – 54

Every planar graph G without cycles of lengths 4 to 8 is 3-choosable.

Given a list L for a graph G, the vertex set of the auxiliary graph $H = H(G, L)$ is $\{(v, c) : v \in V(G) \text{ and } c \in L(v)\}$, and two distinct vertices (v, c) and (v', c') are adjacent in H if and only if either $c = c'$ and $vv' \in E(G)$, or $v = v'$.

G has an L-coloring if and only if the independence number of H is $|V(G)|$.



The definition of DP-coloring

Let G be a graph. A *cover* of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$, satisfying the following conditions.

(1) For each $v \in V(G)$, $H[L(v)]$ is a complete graph.

(2) For each $uv \in E(G)$, the edges between $L(u)$ and $L(v)$ form a matching (possibly empty).

(3) For each distinct $u, v \in V(G)$ with $uv \notin E(G)$, no edges of H connect $L(u)$ and $L(v)$.

An (L, H)-*coloring* of G is an independent set $I \subseteq V(H)$ of size $|V(G)|$. The *DP-chromatic number*, $\chi_{DP}(G)$, is the minimum k such that G has an (L, H)-coloring for each choice of (L, H) with $|L(v)| \geq k$ for all $v \in V(G)$.

f -painting game

Given a graph G and a mapping $f : V(G) \to N$. The f -painting game on G is played by two players: Lister and Painter. Initially, all vertices are uncoloured and each vertex v has $f(v)$ tokens. In the ith step, Lister marks a non-empty subset L_i of uncoloured vertices and takes away one token from each marked vertex. Painter chooses an independent set X_i contained in L_i and colours vertices in X_i by colour i. If at the end of some step, there is an uncoloured vertex v with no tokens left, then Lister wins the game. Otherwise, at some step, all vertices are coloured and Painter wins the game.

f-paintable and the paint number

Suppose $f : V(G) \to N$. We say G is f-paintable if Painter has a winning strategy in the f -painting game on G. We say G is s-paintable for a positive integer s if G is f -paintable for the constant function $f \equiv s$. The paint number $\chi_p(G)$ (also called the paintability and the on-line choice number) of G is the least integer s for which G is s-paintable.

Ming Han, Xuding Zhu, Locally planar graphs are 5-paintable, Discrete Mathematics 338 (2015) 1740 – 1749.

Every graph embedded in a fixed surface with sufficiently large edge-width is 5-paintable.

Ming Han, Xuding Zhu, European Journal of Combinatorics 54 (2016) 35 – 50.

Locally planar graphs are 2-defective 4-paintable.

Adaptably k-coloring (适应点染色)

定义

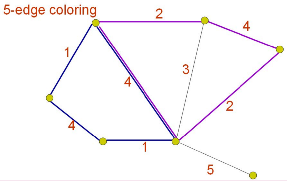
Let G be a graph, and let $F: E(G) \to N$ be a coloring of the edges of G (not necessarily proper). A vertex k-coloring $c: V(G) \to \{1, \cdots, k\}$ of the vertices of G is *adapted* to F if for every $uv \in E(G)$, $c(u) \neq c(v)$ or $c(v) \neq F(uv)$. In other words, the same color never appears on an edge and both its endpoints. If there is an integer k such that for any edge coloring F of G, there exists a vertex k-coloring of G adapted to F, we say that G is adaptably k-colorable. The smallest k such that G is adaptably k-colorable is called the *adaptable chromatic number* of G, denoted by $\chi_{ad}(G)$.

显然：$\chi_{ad}(G) \leq \chi(G)$.

J Graph Theory 62: 127 – 138, 2009

Every K_5-minor-free graph is adaptably 4-choosable;
Every triangle-free planar graph is adaptably 3-choosable.

An edge k-coloring of a graph G is a mapping ψ from $E(G)$ to the set of colors $1, 2, ..., k$ such that any two adjacent edges have different colors. The *edge chromatic number* of a graph G, denoted by $\chi'(G)$, is the smallest integer k such that G has an edge k-coloring.

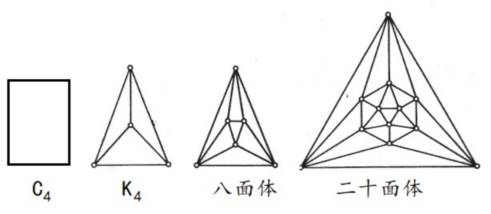


Vizing's Theorem, 1964

For every graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.
A graph G is said to be of *class* 1 if $\chi'(G) = \Delta$, and of *class* 2 if $\chi'(G) = \Delta + 1$.

Four Coloring Problem

For every planar graph G, $\chi(G) \leq 4 \iff$ For every simple 2-edge-connected 3-regular planar G, $\chi'(G) = 3$.



If C_4, K_4, the octahedron, and the icosahedron have one edge subdivided each, class 2 planar graphs are produced for $\Delta \in \{2, 3, 4, 5\}$. Vizing[1] proved that every planar graph with $\Delta \geq 8$ is of class 1 and then posed the following conjecture.

Conjecture 1: 平面图边染色猜想

Every planar graph with $\Delta \geq 6$ is of class 1.

[1] Critical graphs with given chromatic class, Diskret. Analiz. 5 (1965) 9 - 17.

Conjecture 1 is true for planar graphs.

- $\Delta = 7$[a] and [b];
- $(\Delta, g) \in \{(5,4),(4,5),(3,8)\}$, where g is the girth of G[c];
- $\Delta = 6$ and any vertex is incident with at most three triangles[d];
- $\Delta \geq 5$ and any vertex is incident with at most one triangle[e];
- $\Delta = 6$ and G contains no chordal k-cycles for some $k \in \{3,4,5,6,7\}$[f].

[a] L. M. Zhang, Graphs Combin. 16 (2000), 467-495.
[b] Sanders and Y. Zhao J. Combin. Theory Ser B 83 (2001), 202-212
[c] Fiorini and R.J. Wilson, Research Notes in Mathematics, 16, 1977
[d] Wang and Xu, Disc. Appl. Math. 161(2013), 307-310
[e] 陈永珠,王维凡,浙江师范大学学报，30:4(2007),416-420
[f] 倪伟平, 南京师大学报, 34:3(2011), 19-24

Some related papers: [Disc Math, 306(2006), 1440-1445.] [Disc Math, 190(1998), 107-114.] [Congr. Numer. 136 (1999), 201-205.] [Graphs Combin. 19(2003), 393-401.] [Theor Comp Sci, 385:1-3(2007), 71-77.] [Disc Math. 263 (2003), 339-345.]

A graph G is said to be edge f-choosable if, whenever we give lists A_x of $f(x)$ colors to each element $x \in E(G)$, there exists a proper edge coloring of G where each edge is colored with a color from its own list. If $f(x) = k$ for every element $x \in E(G)$, we said G is edge k-choosable. The list edge chromatic number $\chi'_{list}(G)$ is the smallest integer k such that G is edge k-choosable.

Vizing, Metody Diskret. Analiz 29 (1976) 3 – 10

Conjecture 3. Every graph satisfies $\chi'_{list} \leq \Delta + 1$.

The case $\Delta = 3$ was settled in Vizing and, independently, Erdos, Rubin, Taylor [Congr. Numer. 26 (1979) 125 – 157] by proving the choosability version of the Brooks Theorem. the case $\Delta = 4$ is due to Juvan, Mohar, and Skrekovski [Combin. Probab. Comput. 7(1998) 181 – 188].

Borodin, *etc*, J. Comb. Theory(B), 71(1997), 184-204.

Conjecture 4. For any graph G, $\chi'_{list}(G) = \chi'(G)$ and $\chi''_{list}(G) = \chi''(G)$.

Some results on a planar graph with $\chi'_{list} \le \Delta + 1$

- $\Delta \ge 9$[a], $\Delta = 8$[b] and $\Delta \le 4$[c];
- $\Delta \ge 7$且没有(1) 弦7-圈[d], 或(2) 弦6- 圈[e];
- $\Delta \ge 6$且不满足：(1) 相邻3-圈[f], 或(2) 3- 圈与5- 圈相邻[g], 或(3) 弦5-圈[h], 或(4) 弦6-圈[i]
- $\Delta \ge 5$且没有：(1) 弦5-圈和弦4-圈，或(2)弦5-圈和弦6-圈，或(3) 3-圈[j], 或(4) 4-圈[k], 或(4) 5- 圈[l]

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[a] Borodin, Mat. Zametki 48 (6) (1990) 22 – 28
[b] SIAM J. DISCRETE MATH, 29:3(2015), 1735 – 1763
[c] J Graph Theory, 32(1999) 250-262.
[d] ARS Comb. 100(2011), 169-176; DMTCS 15:1, 2013, 101 – 106
[e] Util. Math. 86(2011), 289 – 296; Graphs and Comb (2015) 31:827 – 832
[f] Disc Math 309 (2009) 77 – 84
[g] Disc Math 313 (2013) 575 – 580
[h] Discrete Math. 309(2009) 2233 – 2238
[i] Bull Korean Math Soc 49:2(2012) 359-365
[j] Discrete Mathematics 283 (2004) 289-293
[k] Discrete Mathematics 308(2008) 5789 – 5794
[l] Appl Math Lett, 15(2002) 561-565

Planar graphs on $\chi'_{list} = \Delta$

- $\Delta \geq 12$, or $\Delta \geq 7$ and $g \geq 4$, or $\Delta \geq 5$ and $g \geq 5$, or $\Delta \geq 4$ and $g \geq 6$, or $\Delta \geq 3$ and $g \geq 10$;[a]
- $(\Delta, k) \in \{(7,4), (6,5), (5,8), (4,14)\}$, where k satisfies that G has no cycle of length from 4 to k, where $k \geq 4$. [b]
- $\Delta \geq 8$ and G contains no chordal 5-cycles;[c]
- $\Delta \geq 8$ and G contains no adjacent 4-cycles;[d]
- $\Delta \geq 8$且3-圈和4-圈不邻[e]
- $\Delta \geq 8$且3-圈和5-圈不邻, or $\Delta \geq 7$且两个4^--圈不邻[f]
- $\Delta \geq 7$ and any 4-cycle is not adjacent to 4^--cycles [g]
- $\Delta \geq 6$且没有4-圈和6-圈, or $\Delta \geq 7$且没有5-圈和6-圈[h]

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[a] Borodin, etc, J. Combin. Theory Ser. B 71(1997) 184 - 204

[b] JF Hou, GZ Liu, JS Cai, Theoret. Comput. Sci. 369(2006) 250 - 255

[c] J Comb Optim (2016) 32:188 - 197

[d] J Comb Optim (2016) 31:1013 - 1022

[e] Discrete Mathematics 311 (2011) 2158 - 2163.

[f] Q. Lu, ZK Miao, YQ Wang, Discrete Mathematics 309(2013) 575-580

[g] Acta Math. Sin. (Engl. Ser.) 30(2014), no. 1, 91 - 96.

[h] Information Processing Letters 108 (2008) 347 - 351

我们最近得到的几个结果

L N Hu, H M Song, J L Wu, A note on list edge coloring of planar graphs without adjacent short cycles, ARS Comb.2019.01

(1) A planar graph G is edge-$(\Delta(G)+1)$-choosable if any 4-cycle is not adjacent to a 3-cycle.

(2) If G is a planar graph with $\Delta(G) \geq 6$ and has no adjacent 4-cycles, then $\chi'_{list}(G) \leq \Delta(G)+1$.

L N Hu, L Sun, J L Wu, List edge coloring of planar graphs without 6-cycles with two chords, DMGT, to appear

If G is a planar graph without 6-cycles with two chords, then G is edge-k-choosable, where $k = \max\{7, \Delta(G)+1\}$, and is edge-$t$-choosable, where $t = \max\{9, \Delta(G)\}$.

H Y Wang, J L Wu, List edge coloring of planar graphs without 6-cycles with three chords, J Comb Optim (2018) 35:555 - 562

Let G be a planar graph in which contains no 6-cycles with three chords or G be a f_5-free planar graph. Then G is edge-k-choosable, where $k = \max\{8, \Delta(G)+1\}$, and is edge-$t$-choosable, where $t = \max\{10, \Delta(G)\}$.

我们还获得了一个7-圈不含三条弦的结果.

Total Coloring(全染色)

一个图的全染色是指对图G的点和边都染色使得相邻和相关联的元素之间都染不同的颜色，图G 的全色数,$\chi''(G)$, 是指图G存在一个全染色所用的最少的颜色数.

A well-known conjecture

Conjecture. For any simple graph G, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$.

Theorem

For a planar graph G, $\chi''(G) \leq \Delta + 2$, if one of the following conditions hold.

- $\Delta \leq 5$ *or* $\Delta \geq 7$;
- $\Delta = 6$ *and* $v_5^4 + 2(v_5^{5+} + v_6^4) + 3v_6^5 + 4v_6^{6+} < 24$, *where* v_n^k *represents the number of vertices of degree* n *which lie on* k *distinct* 3*-cycles;* [Graphs and Comb. 30(2014), 377-388]
- $\Delta = 6$ *and without* 4-, 5-, *or* 6*-cycles with chords* [Hou and Liu]
- $\Delta = 6$ *and two cycles of length at most* 5 *are not adjacent.*[Wu and Fang]

对平面图G, $\chi''(G) = \Delta + 1$如果下列条件之一成立:

(1) $\Delta \geq 14, 12, 11, 10$ and finally 9;

(2) $\Delta \geq 8$ and for every vertex $x \in V(G)$, there is an integer $k \in \{3, 4, 5, 6, 7, 8\}$ such that x is incident with at most one cycle of length k ;

(3) $\Delta \geq 8$ and for each vertex x, there are two integers $i, j \in \{3, 4, 5, 6\}$ such that any two cycles of length i and j, which contain x, are not adjacent;

(4) $\Delta \geq 8$ and G is an F_5-free planar graph;

(5) $\Delta \geq 8$ and G contains no 5-cycles with two chords;

(6) $\Delta \geq 8$ and G contains no adjacent chordal 5-cycles;

(7) $\Delta \geq 8$ and G contains no adjacent chordal 7-cycles;

(8) $\Delta \geq 8$ and G contains no 6-cycles with two chords or adjacent chordal 6-cycles;

(9) $\Delta \geq 8$ and G contains no 7-cycles with three chords;

(10) $\Delta \geq 7$ and for every vertex $x \in V(G)$, there is an integer $k \in \{3, 4, 5, 6, 7, 8\}$ such that x is incident with no cycles of length k;

(11) $\Delta \geq 7$ and every vertex v has an integer $k_v \in \{3, 4, 5, 6\}$, such that v is not in any k_v-cycle;

如果满足下列条件之一，则对平面图G有$\chi''(G) = \Delta + 1$:

(12) $\Delta \geq 7$ and G contains no intersecting 3-cycles, or adjacent 4-cycles, adjacent 5-cycles, or intersecting 6-cycles;

(13) $\Delta \geq 7$ and G contains no chordal i-cycles($i = 5, 6$, or 7);

(14) $\Delta \geq 7$ and no 3-cycle is adjacent to a cycle of length less than 6;

(15) $\Delta \geq 6$ and G contains no 5-cycles and 6-cycles, or $\Delta \geq 5$ and G contains no 4-cycles and 6-cycles;

(16) $\Delta(G) \geq 6$, G contains no intersecting 4-cycles and G contains no intersecting 3-cycles,or 5-cycles,or 6-cycles;

(17) $\Delta \geq 6$ and G contains no 4-cycles;

(18) $\Delta \geq 6$ and G contains no adjacent 4^--cycles;

(19) $(\Delta, g) \in \{(7, 4), (5, 5), (4, 6), (3, 10)\}$, where g is the girth of G;

(20) $(\Delta, k) \in \{(7, 4), (6, 5), (5, 7), (4, 14)\}$, where G has no cycle of length from 4 to k, where $k \geq 4$;

(21) $(\Delta, k) \in \{(6, 4), (5, 5), (4, 11)\}$, where G contains no intersecting 3-cycles and G has no cycle of length from 4 to k.

A graph G is said to be total f-choosable if, whenever we give lists A_x of $f(x)$ colors to each element $x \in V(G) \cup E(G)$, there exists a proper total coloring of G where each element is colored with a color from its own list. If $f(x) = k$ for every element $x \in V(G) \cup E(G)$, we said G is total k-choosable. The list total chromatic number $\chi''_{list}(G)$ is the smallest integer k such that G is total k-choosable.

Borodin, *etc*, J. Comb. Theory(B), 71(1997), 184-204.

Conjecture 4. For any graph G, $\chi'_{list}(G) = \chi'(G)$ and $\chi''_{list}(G) = \chi''(G)$.

此猜想对二分图是成立的。

Some results on a planar graph with $\chi''_{list}(G) \le \Delta + 2$

- $\Delta \ge 9$[a];
- $\Delta \ge 7$且(1) 没有弦7-圈[b], 或(2) F_5-free [c], 或(3) 每个3- 圈至多与其他一个3- 圈相邻[d];
- $\Delta \ge 6$且不满足：(1) 弦6- 圈[e], 或(2) 3- 圈与5- 圈相邻[f], 或(3) 3- 圈与4- 圈相邻[g];
- $\Delta \ge 5$且没有：(1) 弦5-圈和弦4-圈, 或(2)弦5-圈和弦6-圈, 或(3) 3-圈或4- 圈[h], 或(4) 5- 圈[i]

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[a] LNCS 4489 (2007) 320 – 328.

[b] ARS Comb. 100(2011), 169-176; DMTCS 15:1(2013), 101 – 106

[c] J Comb Optim, to appear.

[d] 山大学报,2009年10期

[e] Bull Korean Math Soc 49:2(2012) 359-365

[f] Disc Math 313 (2013) 575 – 580

[g] Discrete Mathematics 311 (2011) 2158 – 2163

[h] LNCS 4489 (2007) 320 – 328

[i] Appl Math Lett, 15(2002) 561-565

Planar graphs on $\chi''_{list}(G) = \Delta+1$

- $\Delta \geq 12$;[a]
- $(\Delta, k) \in \{(7,4), (6,5), (5,8), (4,14)\}$, where k satisfies that G has no cycle of length from 4 to k, where $k \geq 4$. [b]
- $\Delta \geq 8$ and G contains no chordal 5-cycles;[c]
- $\Delta \geq 8$ and G contains no adjacent 4-cycles;[d]
- $\Delta \geq 8$且3-圈和4-圈不邻[e]
- $\Delta \geq 8$且3-圈和5-圈不邻, or $\Delta \geq 7$且两个4^--圈不邻[f]
- $\Delta \geq 7$ and any 4-cycle is not adjacent to 4^--cycles [g]
- $\Delta \geq 6$且没有4-圈和6-圈, or $\Delta \geq 7$且没有5-圈和6-圈[h]

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[a] Borodin, *etc*, J. Combin. Theory Ser. B 71(1997) 184 – 204
[b] JF Hou, GZ Liu, JS Cai, Theoret. Comput. Sci. 369(2006) 250 – 255
[c] J Comb Optim (2016) 32:188 – 197
[d] J Comb Optim (2016) 31:1013 – 1022
[e] Discrete Mathematics 311 (2011) 2158 – 2163.
[f] Q. Lu, ZK Miao, YQ Wang, Discrete Mathematics 309(2013) 575-580
[g] Acta Math. Sin. (Engl. Ser.) 30(2014), no. 1, 91 – 96.
[h] Information Processing Letters 108 (2008) 347 – 351

进一步研究和思考的问题

(1) 证明满足如下条件的平面图的列表色数至多为4，或列表点荫度至多为2：4-圈不相交、6- 圈不交、不含弦5- 圈、3圈与5- 圈不相邻等。

(2) 最近我们证明了：$\Delta \geq 7$且7-圈至多有2条弦或6-圈至多有2条弦的平面图的列表边色数至多为$\Delta + 1$。这样我们还可以考虑如下条件: (1) $\Delta \geq 7$且5-圈至多有1 条弦; (2) $\Delta \geq 7$且弦k- 圈不相邻($k = 4, 5, 6, 7$); (3) $\Delta \geq 6$且弦k-圈不相交($k = 4, 5, 6, 7$)。当然，列表边色数等于最大度、或列表线性荫度、或列表全色数的条件可以更多。

(3) 证明：(1) 最大度为8的平面图的列表全色数至多为10; (2) 最大度为11的平面图的列表全色数等于12；(3) 考虑relaxed, separated, different的情况。

(4) 证明：最大度为6的图或平面图的全色数至多是8。

(5) 证明：最大度为7的图的线性荫度是4;

(6) 已知1-平面图的点色数至多为6，无圈点色数≤ 20, 那么它的列表无圈点色数、无圈列表边色数的上界是多少？对NIC-和IC-平面图的结果又是多少？

(7) 已知$\Delta \geq 9$的平面图的均匀色数$\leq \Delta$, 那么1-平面图的最大度至少为多少的时候也有此结果呢？列表均匀色数呢？

(8) 最近我们证明了：(1) 对任意的$k \geq 12$, 所有的平面图都是可均匀k-边染色的; (2) 对任意的$k \geq 21$, 所有的1- 平面图都是可均匀k-边染色的。那么对嵌入到欧拉示性数小于0 的曲面上的图，它的均匀边色数又是一个什么结果？

(9) 2016年在European Journal of Combinatorics上发表一篇题目为“Choosability in signed planar graphs”的文章，我们完全可以考虑符号图的其它染色。

(10) 把以上所得到结果的条件用于其它的染色，如无圈点染色、无圈边染色、无圈全染色、均匀点染色、均匀边染色、均匀点荫度、邻点可区别的各种染色等，但是要注意结果的包含关系;

谢谢各位的聆听！