The inducibility of oriented stars

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Base on joint works with Qilin, Dong
and with Ping Hu, Jie Ma, Sergey Norin

Terminology

- $\mathcal{C}(H;G)$: the number of induced subgraphs of G isomorphic to G.
- *The induced density of H in G:*

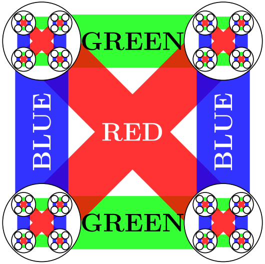
$$i(H;G) = \frac{\mathcal{C}(H;G)}{\binom{|G|}{|H|}}$$

- $i(H;n)$: $\max_{|G|=n} i(H;G)$.
- *The inducibility of H:* $i(H) = \lim_{n\to\infty} i(H,n)$.

Rainbow-triangle

Theorem (Balogh-Hu-Lidický-Pfender 2016)

Maximum density of rainbow-triangle in 3-edge-coloring of K_n is achieved by the following graph:

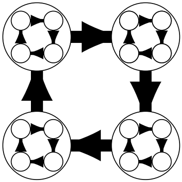


Inducibility of $S_{1,1}$

- $\mathbf{S_{i,j}}$: the oriented star with i arcs oriented out from the center and j arcs oriented into the center.

Theorem (Hadlky-Kral-Norin)

Maximum induced density of $S_{1,1}$ in a directed graph is achieved by the following graph:



Inducibility of $S_{0,k}$

Question: What is the maximum induced density for $S_{i,j}$?

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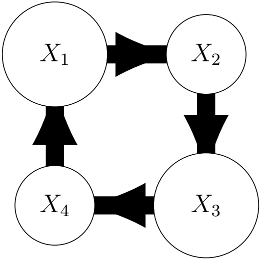
Theorem (Falgas-Ravry, Vaugha (k=1,2), Huang ($k \geq 3$))

Maximum induced density of $S_{0,k}$ in a directed graph is achieved by a unbalanced blow-up of a transitive tournaments.

Inducibility of $S_{2,2}$

Theorem (Dong, W.)

Maximum density of $S_{2,2}$ in a directed graph is achieved by an orientation of a complete bipartite graph $D[A, B]$, such that for each vertex, the difference between in degree and out degree is at most 1, and $\frac{|A|}{|B|} = 2 + \sqrt{3}$. In particular, $\pi_{S_{2,2}}(\emptyset) = \frac{5}{32} = 0.15625$.



Sketch of proof

By Flag Algebraic, we have the following

Lemma

$\pi_{S_{2,2}}(\emptyset) \leq 0.15625 + \epsilon$ for some small ϵ.

Use Flag Algebraic again with $\pi_{S_{2,2}} \geq 0.15625$, we have

Lemma

For extremal graph G, we have $d(T_1; G) + d(T_2; G) + d(T_3; G) \leq \epsilon'$ for some small $\epsilon' > 0$.

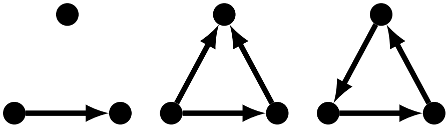


Figure: T_1, T_2 and T_3.

Proposition

For extremal graph G, there exists a partition (A, B), such that after changing at most $o(n^2)$ edges, it will be an orientation of complete bipartite graph. Furthermore, it can be change into the extremal graph that we want.

We define two types of bad vertices:

- Many "bad" edges.
- Gap between in degree and out degree are big.

Proposition

There are $o(n)$ bad vertices.

Proposition

There is no more "bad" edges after removing "bad" vertices".

Proposition

Fit the "bad" vertices back to the A or B.

Inducibility of $S_{k,l}$

Theorem (Hu, Ma, Norin, W.)

Let k and l be positive integers with $k + l \geq 10$. Then when $k = \ell$,

$$i(S_{k,\ell}) = \frac{(k+\ell+1)!}{2^{k+\ell}k!\ell!} \cdot \max_{\alpha \in [0,1]} \left\{ \alpha(1-\alpha)^{k+\ell} + (1-\alpha)\alpha^{k+\ell} \right\};$$

and when $k \geq \ell + 1$, $i(S_{k,\ell})$ is equal to

$$\frac{(k+\ell+1)!}{k!\ell!} \max_{\alpha,d} \{ \alpha(1-\alpha)^{k+\ell} d^k (1-d)^\ell + (1-\alpha)\alpha^{k+\ell}(1-d)\frac{(k-1)^{k-1}\ell^\ell}{(k+\ell-1)^{k+\ell-1}} \}$$

where the maximum is over all possible pairs $(\alpha, d) \in [0, \frac{1}{2}] \times [\frac{\ell}{k+\ell}, \frac{k}{k+\ell}]$.

Inducibility of $S_{k,l}$

With α, d be the pair achieved maximum in last theorem, we have $\alpha \approx \frac{1}{k+l+1}$, and $d \approx \frac{k}{k+l}$

Theorem (Hu, Ma, Norin, W.)

Given $k + l \geq 10$, and n sufficient large, the maximum induced density $i(K_{k,l})$ is achieved by an orientation of complete bipartite graph $D(X, Y)$, such that

- $|X| = \alpha n, |Y| = (1 - \alpha)n$.
- *For each vertex x in X, $d|Y|$ of arcs are oriented from x to Y.*
- *$Y = Y_1 \dot\cup Y_2$, for each vertex in Y_2, all arcs are oriented from X, and for each vertex in Y_1, $\frac{l}{k+l}$ of arcs oriented into it.*

Sketch of proof

Claim

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Lemma

$|X| \approx \alpha n, |Y| \approx (1-\alpha)n$, for each vertex x in X, it has approximate $(1-\alpha)n$ arcs, among which roughly d of them are oriented out from x. And each vertex y in Y has approximate αn arcs.

Sketch of proof

- Step 1: Each vertex in X has in/out degree very close to extremal graph;
- Step 2: There are not many edges in X and in Y;
- Step 3: The edges between X and Y are complete;
- Step 4: Most of the Star centers in Y are center in a set Y_2;
- Step 5: There is no edge in Y;
- Step 6: There is no edge in X;
- Step 7: Determine the sizes by calculations.

Thank you for your attention!