Bouns and Constructions for Optimal $(v, \{3, 4, 5\}, \Lambda_a, 1)$-OOCs

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Introduction

Optical Orthogonal Codes (OOCs)

Optical orthogonal codes (OOCs) were introduced by Salehi, as signature sequences to facilitate multiple access in optical fibre networks [1-2].

[1] J. A. Salehi, Code division multiple access techniques in optical fiber networks-Part I Fundamental Principles, IEEE Trans. Commun., 37(1989), 824-833.

[2] J. A. Salehi and C. A. Brackett, Code division multiple access techniques in optical fiber networks-Part II Systems performance analysis, IEEE Trans. Commun., 37(1989), 834-842. 1989.

Optical Orthogonal Codes (OOCs)

OOCs have found wide range of applications such as mobile radio, frequency-hopping spread-spectrum communications, radar, sonar, collision channel without feedback and neuromorphic network [3-7].

[3] F. R. K. Chung, J. A. Salehi and V. K. Wei, Optical orthogonal codes: Design, analysis, and applications, IEEE Trans. Inform. Theory, 35(1989), 595-604.

[4] S. W. Golomb, Digital communication with space application, Los Altos, CA: Penisula, 1982.

[5] J. L. Massey and P. Mathys, The collision channel without feedback, IEEE Trans. Inform. Theory, 31(1985), 192-204.

[6] J. A. Salehi, Emerging optical code-division multiple-access communications systems, IEEE Network, 3(1989), 31-39.

[7] M. P. Vecchi and J. A. Salehi, Neuromorphic networks based on sparse optical orthogonal codes, in Neural Information Processing Systems-Natural and Synthetic, New York: Amer. Inst. Phys., 1988, 814-823.

$(v, k, \lambda_a, \lambda_c)$-OOC

A $(v, k, \lambda_a, \lambda_c)$-OOC $\mathcal{C}$ is a family of $(0, 1)$ sequences of length v and weight k satisfying the following two properties:

- *Auto-correlation:* For any $\mathbf{x} = (x_0, x_1, \ldots, x_{v-1}) \in C$, and any integer τ, $0 < \tau < v$,

$$\sum_{t=0}^{v-1} x_t x_{t \oplus \tau} \leq \lambda_a,$$

  where the summation is carried out by treating binary symbols as reals.

- *Cross-correlation:* Similarly, for any $\mathbf{x} \neq \mathbf{y}$, $\mathbf{x} = (x_0, x_1, \ldots, x_{v-1}) \in C$, $\mathbf{y} = (y_0, y_1, \ldots, y_{v-1}) \in C$, and any integer τ,

$$\sum_{t=0}^{v-1} x_t y_{t \oplus \tau} \leq \lambda_c.$$

$(v, k, \lambda_a, \lambda_c)$-OOC

(v, k, λ)-OOC $\mathcal{C}$ A $(v, k, \lambda_a, \lambda_c)$-OOC with property that $\lambda_a, \lambda_c = \lambda$.

Example 1 A $(37, 4, 1)$-OOC with 3 codewords.

1101000000000000000000001000000000000,
1000100001000001000000000000000000000,
1000000100000000010000000100000000000.

Most existing work on OOC's have assumed that all codewords have the same weight.

$(v, k, \lambda_a, \lambda_c)$-OOC

Example 2 A $(17, 4, 2, 1)$-OOC with 2 codewords.

11001100000000000, 10100000101000000.

Let $\Phi(n, k, \lambda_a, \lambda_c)$=max$\{|C| : C$ is an $(n, k, \lambda_a, \lambda_c)$-OOC$\}$.

$\Phi(n, k, 1) \leq \lfloor \frac{n-1}{k(k-1)} \rfloor$.

$\Phi(n, 3, 2, 1)(\approx) \leq \frac{n-1}{4}$.

$\Phi(n, 4, 2, 1)(\approx) \leq \frac{n-1}{8}$.

$\Phi(n, 5, 2, 1)(\approx) \leq \frac{n-1}{12}$.

$(v, k, \lambda_a, \lambda_c)$-OOC

For $k = 3, 4$

[8] T. Baicheva and S. Topalova, Optimal $(v, 4, 2, 1)$ optical orthogonal codes with small parameters, J. Combin. Des. 20(2012), 142-160.

[9] M. Buratti, K. Momihara, and A. Pasotti, New results on optimal $(v, 4, 2, 1)$ optical orthogonal codes, Des. Codes Cryptogr. 58(2011), 89-109.

[10] K. Momihara and M. Buratti, Bounds and constructions of optimal $(n, 4, 2, 1)$ optical orthogonal codes, IEEE Trans. Inform. Theory 55(2009), 514-523.

[11] X. Wang and Y. Chang, Further results on optimal $(v, 4, 2, 1)$-OOCs, Discr. Math. 312(2012), 331-340.

For $k = 5$

[12] M. Buratti, A. Pasotti, and D. Wu, On optimal $(v, 5, 2, 1)$ optical orthogonal codes, Des. Codes Cryptogr., 682013349-371.

Variable-Weight OOCs

VWOOCs

In 1996, G-C Yang introduced multimedia optical CDMA communication system employing variable-weight OOCs [14]. The variable-weight property of the OOCs enables the system to meet multiple QoS requirement. Variable-weight OOCs have attracted much attention [13-15].

[13] G. C. Yang, Variable-weight optical orthogonal codes for CDMA networks with multiple performance requirements, IEEE Trans. Commun., 44(1996), 47-55.

[14] I. B. Djordjevic, B. Vasic, J. Rorison, Design of multiweight unipolar codes for multimedia optical CDMA applications based on pairwise balanced designs, J. Lightwave Technol., 21(2003), 1850-1856.

Definition of VWOOC

$W = \{w_1, ..., w_r\}$ an ordering of a set of r integers greater than 1, $\Lambda_a = (\lambda_a^{(1)}, ..., \lambda_a^{(r)})$ an r-tuple (*auto-correlation sequence*) of positive integers, λ_c a positive integer (*cross-correlation parameter*), and $Q = (q_1, ..., q_r)$ an r-tuple (*weight distribution sequence*) of positive rational numbers whose sum is 1.

Definition of a VWOOC

Definition A $(v, W, \Lambda_a, \lambda_c, Q)$ variable-weight optical orthogonal code $\mathcal{C}$, or $(v, W, \Lambda_a, \lambda_c, Q)$-OOC, is a collection of binary v-tuples such that the following three properties hold:

Weight Distribution Every v-tuple in $\mathcal{C}$ has a Hamming weight contained in the set W; furthermore, there are exactly $q_i|\mathcal{C}|$ codewords of weight w_i, $1 \leq i \leq r$.

Periodic Auto-correlation For any $\mathbf{x} = (x_0, x_1, \ldots, x_{v-1}) \in \mathcal{C}$ with Hamming weight $w_i \in W$, and any integer τ, $0 < \tau < v$,

$$\sum_{t=0}^{v-1} x_t x_{t \oplus \tau} \leq \lambda_a^i,$$

where the summation is carried out by treating binary symbols as reals.

Definition of a VWOOC

Periodic Cross-correlation For $\mathbf{x} \neq \mathbf{y}$, $\mathbf{x} = (x_0, x_1, \ldots, x_{v-1}) \in \mathcal{C}$, $\mathbf{y} = (y_0, y_1, \ldots, y_{v-1}) \in C$, and any integer τ,

$$\sum_{t=0}^{v-1} x_t y_{t \oplus \tau} \leq \lambda_c.$$

[15] M. Buratti, Y. Wei, D. Wu, P. Fan, and M. Cheng, Relative difference families with variable block sizes and their related OOCs, IEEE Trans. Inform. Theory. 57(2011), 4488-4496.

Definition of a VWOOC

$(v, W, \lambda_a, \lambda_c, Q)$-OOC: $\lambda_a^{(i)} = \lambda_a$ for every i.

(v, W, λ, Q)-OOC: $\lambda_a = \lambda_c = \lambda$.

Q is *normalized*: Q is written in the form $Q = (\frac{a_1}{b}, \ldots, \frac{a_r}{b})$ with $\gcd(a_1, \ldots, a_r) = 1$.

A $(v, k, \lambda_a, \lambda_c)$-OOC (constant-weight OOC) is a $(v, W, \Lambda_a, \lambda_c, Q)$-OOC with $W = \{k\}$, $\Lambda_a = \{\lambda_a\}$, and $Q = \{1\}$. A (v, k, λ)-OOC is a $(v, k, \lambda_a, \lambda_c)$-OOC with $\lambda_a = \lambda_c = \lambda$.

Definition of a VWOOC

Example 3 A $(80, \{4, 5\}, 1, (1/2, 1/2))$-OOC with 4 codewords.

1001000000000000000000001000000000000000000000100000000001000000000000000000000

1000100000000000101000000000000000000000000000000000000000000000000100000000000

1100000100000000000000000000000000000001000000000000000000000000000000000000000

1000000000000000000100000000100000000000000000000000000000000000000000001000000

Definition of a VWOOC

We give an example to show that variable-weight OOCs can generate larger code size than that of constant-weight OOCs. Let

$$
\begin{aligned}
\mathcal{C}^{(1)} &= \{1101000100000000100000000000000000000000\}; \\
\mathcal{C}^{(2)} &= \{1101000001000000000000000000100000000000, \\
&\quad 1000000000010000001000000000000000010000, \\
&\quad 1000000000000000000000000010001000000000\}; \\
\mathcal{C}^{(3)} &= \{1101000100000000100000000000000000000000, \\
&\quad 1000010000000000000100000000000000000000, \\
&\quad 1000000010000000000000000000001000000000, \\
&\quad 1000000000000000010000000000010000000000\}.
\end{aligned}
$$

Then $\mathcal{C}^{(1)}$ is an optimal $(40, 5, 1)$-OOC (only one codeword), $\mathcal{C}^{(2)}$ is an optimal $(40, \{3, 4, 5\}, 1, (1/3, 1/3, 1/3))$-OOC (three codewords), $\mathcal{C}^{(3)}$ is an optimal $(40, \{3, 5\}, 1, (3/4, 1/4))$-OOC (four codewords).

Known Results on $(n, W, 1, Q)$-OOCs

Yang [16]: $(n, \{s, s+1\}, 1, 1, Q)$-OOCs, $(n, \{s, s+1\}, 2, 1, Q)$-OOCs, $(n, \{2s, s\}, (2, 1), 1, Q)$-OOCs, $(n, \{2s, s\}, 2, 1, Q)$-OOCs. Some of them are optimal, and $|W| = 2$.

Wu et. al: Optimal $(n, W, 1, Q)$-OOCs for $W \in \{\{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{3, 4, 5\}, \{3, 4, 6\}, \{3, 4, 5, 6\}\}$. IEEE IT, DM, JCD, AMC.

[17] G. C. Yang, Variable-weight optical orthogonal codes for CDMA networks with multiple performance requirements, IEEE Trans. Commun., 44(1996), 47-55.

$(n, W, \Lambda_a, 1, Q)$-OOCs

$(n, W, \Lambda_a, 1, Q)$-OOCs

Let $\binom{Z_n}{k}$ be the set of all k-subsets of Z_n, the residue ring of integers modulo n. By identifying codewords in an $(n, W, \Lambda_a, \lambda_c, Q)$-OOC $\mathcal{C}$ with subsets of Z_n representing the indices of the nonzero positions, one can get the following convenient definition of variable-weight OOC from set point of view.

Definition An $(n, W, \Lambda_a, \lambda_c, Q)$-OOC is a set $\mathcal{C}$ of subsets (*codeword-sets*) of Z_n with sizes (*weights*) from W satisfying the following properties:

Weight distribution property the ratio of codeword-sets of $\mathcal{C}$ of weight w_i is q_i:

$$\left|\mathcal{C} \cap \binom{Z_n}{w_i}\right| = q_i|\mathcal{C}| \quad \text{for } 1 \le i \le r;$$

$(n, W, \Lambda_a, 1, Q)$-OOCs

Auto-correlation property any two distinct translates of a codeword-set of weight w_i share at most $\lambda_a^{(i)}$ elements:

$$|C \cap (C+t)| \le \lambda_a^{(i)} \quad \forall\, C \in \mathcal{C} \cap \binom{Z_n}{w_i},\ \forall\, t \in Z_n \setminus \{0\}; \quad (1)$$

Cross-correlation property any two translates of two distinct codeword-sets share at most λ_c elements:

$$|C \cap (C'+t)| \le \lambda_c \qquad \forall\, \{C, C'\} \in \binom{\mathcal{C}}{2},\ \forall\, t \in Z_n. \quad (2)$$

[18] K. Momihara and M. Buratti, Bounds and constructions of optimal $(n, 4, 2, 1)$ optical orthogonal codes, IEEE Trans. Inform. Theory 55(2009), 514-523.
[19] H. Zhao, D. Wu, R. Qin, Further results on balanced $(n, \{3, 4\}, \Lambda_a, 1)$-OOCs, Discr. Math. 337(2014) 87-96.

Known Results on $(n, W, \Lambda_a, 1, Q)$-OOCs

Bounds and constructions of optimal $(n, W, \Lambda_a, 1, Q)$-OOCs were obtained for the following cases:

$$W = \{3, 4\},\ \Lambda_a = (1, 2), (2, 1), (2, 2), (2, 3)$$

[20] H. Zhao, D. Wu, R. Qin, Further results on balanced $(n, \{3, 4\}, \Lambda_a, 1)$-OOCs, Discr. Math. 337(2014) 87-96.

[21] Bichang Huang, Yueer Wei, Dianhua Wu, Bounds and constructions for optimal variable-weight OOCs with unequal auto- and cross-correlation constraints, Utilitas Mathematica, to appear.

[22] Maoxing Zhu, Bounds and Constructions for Optimal $(n, \{3, 4\}, \Lambda_a, 1, Q)$-OOCs, Master Degree Thesis, Guangxi Normal University, 2016.

Known Results on $(n, W, \Lambda_a, 1, Q)$-OOCs

$W = \{3, 5\}$, $\Lambda_a = (1, 2), (2, 1), (2, 2)$

[23] Wei Li, Huangsheng Yu, Dianhua Wu, Bounds and constructions for optimal $(n, \{3, 5\}, \Lambda_a, 1, Q)$-OOCs, Discrete Math., 339(2016), 21-32.

$W = \{4, 5\}$, $\Lambda_a = (1, 2), (2, 1), (2, 2)$

One manuscript is in preparing.

$(n, \{3, 4, 5\}, \Lambda_a, 1, Q)$-OOCs

$\Lambda_a = (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)$

[24] Huangsheng Yu, Shujuan Dang, Dianhua Wu, Bounds and constructions for optimal $(n, \{3, 4, 5\}, \Lambda_a, 1, Q)$-OOCs, IEEE Trans. Inform. Theory, 64(2)(2018), 1361-1367.

$\Lambda_a = (1, 2, 1), (1, 1, 2), (1, 2, 2)$

[25] Huangsheng Yu, Feifei Xie, Dianhua Wu, Hengming Zhao, Further results on optimal $(n, \{3, 4, 5\}, \Lambda_a, 1, Q)$-OOCs, Advances in Mathematics of Communications, 13(2)(2019), 297-312.

$(n, \{3, 4, 5\}, \Lambda_a, 1, Q)$-OOCs

Let $\Phi(n, W, \Lambda_a, 1, Q)$ =max$\{|\mathcal{C}| : \mathcal{C}$ *is an* $(n, W, \Lambda_a, 1, Q)$-OOC$\}$. Suppose that $Q = (\frac{a_1}{b}, \frac{a_2}{b}, \frac{a_3}{b})$ is normalized.

Theorem 1 Let $h = 4a_1 + 8a_2 + 20a_3$, then

$$\Phi(n, \{3, 4, 5\}, (2, 2, 1), 1, Q) \leq \begin{cases} b\lfloor \frac{n-1}{h} \rfloor, & \gcd(n, 28) = 1; \\ b\lfloor \frac{n}{h} \rfloor, & \gcd(n, 28) = 2, 4; \\ b\lfloor \frac{n+1}{h} \rfloor, & \gcd(n, 28) = 7; \\ b\lfloor \frac{n+2}{h} \rfloor, & \gcd(n, 28) = 14, 28. \end{cases}$$

Theorem 2 Let $p \geq 3$ be a prime, then there exists an optimal $(16p, \{3, 4, 5\}, (2, 2, 1), 1, (1/3, 1/3, 1/3))$-OOC.

$(n, \{3,4,5\}, \Lambda_a, 1, Q)$-OOCs

Theorem 3 Let $e = 6a_1 + 8a_2 + 20a_3$, then

$$\Phi(n, \{3,4,5\}, (1,2,1), 1, Q) \leq \begin{cases} b\lfloor \frac{n-1}{e} \rfloor, & \text{if } \gcd(n,14) = 1; \\ b\lfloor \frac{n}{e} \rfloor, & \text{if } \gcd(n,14) = 2; \\ b\lfloor \frac{n+1}{e} \rfloor, & \text{if } \gcd(n,14) = 7; \\ b\lfloor \frac{n+2}{e} \rfloor, & \text{if } \gcd(n,14) = 14. \end{cases}$$

Theorem 4 For any prime $p \geq 3$, there exists an optimal $(17p, \{3,4,5\}, (1,2,1), 1, (1/3, 1/3, 1/3))$-OOC.

Key Points to the Proof of the Theorems

1. Find the equiverlent combinatorial designs.
2. The structures of the Base Blocks (For the upper bounds).
3. Construct the corresponded designs (optimal constructions).

Problems for Further Research

1. Systematic Constructions for $(n, W, \{3, 4, 5\}, 1, Q)$-OOCs.
2. Systematic Constructions for $(n, W, \{3, 4, 5\}, \Lambda_a, 1, Q)$-OOCs..

Thanks for your attention!