The Lagrangian densities of r-uniform hypergraphs

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Outline

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Turán problem

Question

Given an r-uniform hypergraph (r-graph) F and integer n, what is the **maximum** number of edges an r-graph with n vertices can have **without containing** a copy of F as a subgraph?

This number (**Turán number**) is denoted by $ex(n, F)$.

Definition

The **Turán density** of an r-graph F is defined to be

$$\pi(F) = \lim_{n \to \infty} \frac{ex(n, F)}{\binom{n}{r}}.$$

Remark: An argument of Katona, Nemetz, Simonovits implies that such a limit exists.

Turán results

Theorem (Erdős-Stone-Simonovits, -)

Let F be a graph with at least 1 edge. Then

$$ex(n, F) = \frac{1}{2}\left(1 - \frac{1}{\chi(F) - 1}\right)n^2 + o(n^2),$$

where $\chi(F)$ is the chromatic number of F.

Remark: $\pi(F) = 1 - \frac{1}{\chi(F)-1}$.

Question

$\pi(F) =?$ when F is a **bipartite graph**?

Determining the Turán density of hypergraphs is much harder, very few results are known for hypergraphs.

Lagrangian

Definition

G: an r-graph with vertex set $\{1, 2, ..., n\}$, $\vec{x} = (x_1, \ldots, x_n) \in [0, \infty)^n$. Define

$$\lambda(G, \vec{x}) = \sum_{e \in G} \prod_{i \in e} x_i$$

and

$$\lambda(G) = \sup\{\lambda(G, \vec{x}) : x_i \geq 0, x_1 + \cdots + x_n = 1\}.$$

- If $G \subseteq H$ then $\lambda(G) \leq \lambda(H)$
- $|G|/n^r \leq \lambda(G) \leq 1/r!$

Lagrangian

Theorem(Motzkin-Straus, 1965)

If G is a graph in which a largest clique has order t then

$$\lambda(G) = \lambda(K_t^2) = \frac{1}{2}\left(1 - \frac{1}{t}\right).$$

Remark: The obvious generalization of Motzkin and Straus' result to r-uniform hypergraphs is false.

Definition

A r-graph G is **dense** if any proper subgraph H of G satisfies $\lambda(H) < \lambda(G)$.

Remark: A graph is dense $\Leftrightarrow$ it is a complete graph.

Connection between Lagrangian and Turán density

F, G : r-graphs

Definition

$f : V(F) \to V(G)$ is called a **homomorphism** if it preserves edges. G is F**-hom-free** if there is no homomorphism from F to G.

Theorem (Sidorenko, 1989)

$\pi(F)$ is the supremum of $r!\lambda(G)$ over all dense F-hom-free G.

Remark: Let $L(F)$ be the supremum of Lagrangians of all F-free r-graphs then $\pi(F) \le r!L(F)$.

Outline

Lagrangian densities of r-graphs

Definition

Given an r-graph F, define the **Lagrangian density** of F, denoted by $\pi_\lambda(F)$, as

$$\pi_\lambda(F) = r! \sup\{\lambda(G) : G \text{ is } F\text{-free}\}.$$

Question(Hefetz-Keevash, 2012)

Given an r-graph F, what is the number of $\pi_\lambda(F)$?

- Turán type problem
- Hard
- Few results are known

What kind of r-graphs are perfect?

Let $M_t^r = \{e_1, e_2, \ldots, e_t\}$, where $|e_i| = r$ and $e_i \cap e_j = \emptyset$.

Theorem(Hefetz-Keevash, JCTA, 2012)

$$\pi_\lambda(M_2^3) = 3!\lambda(K_5^3) = 12/25.$$

Definition

An r-graph H is **perfect** if $\pi_\lambda(H) = r!\lambda(K^r_{v(H)-1})$.

Remark: M_2^3 is perfect.

Matching

Theorem(Jiang-Peng-Wu, YEUJC, 2018)

$$\pi_\lambda(M_t^3) = 3!\lambda(K_{3t-1}^3) = 12/25.$$

Theorem(Bene Watts-Norin-Yepremyan)

Let $r \in \{4, 5, \dots\}$.

$$\pi_\lambda(M_2^r) = r!\lambda(\mathcal{S}^r) = (1-r)^{r-1}.$$

Theorem(Wu-Peng-Chen, Manuscript)

- $\pi_\lambda(M_t^4) = 4!\lambda(\mathcal{S}_{t-1}^4)$ for $t \in \{2, 3\}$
- $\pi_\lambda(M_t^4) = 4!\lambda(K_{4t-1}^4) = \frac{4!\binom{4t-1}{4}}{(4t-1)^4}$ for $t \in \{4, 5, 6, 7, 8, 9, 10\}$

Linear Path

A **linear path** $P_t^r = \{e_1, e_2, \cdots, e_t\}$, where $|e_i \cap e_{i+1}| = 1$ and $e_i \cap e_j = \emptyset$ for every $i \in [t-1]$ and $j \geq i+2$.

Theorem(Jiang-Peng-Wu, r=3,4, 2018; Wu-Peng, $r = 5$, Manuscript)

Let $r \in \{3, 4, 5\}$,

$$\pi_\lambda(P_2^r) = 4!\lambda(K_{2r-2}^r) = \frac{4!\binom{2r-2}{r}}{(2r-2)^r}.$$

Theorem(Wu-Peng, Submitting)

Let $t = 3$ or 4,

$$\pi_\lambda(P_t^3) = 4!\lambda(K_{2t}^3) = \frac{3!\binom{2t}{4}}{(2t)^3}.$$

Outline

Extension

Definition

The **extension** of an r-graph G, H^G, is obtained from it by adding for every pair of vertices of G, which is not covered by an edge in G, an extra edge containing this pair and $(r-2)$ new vertices.

- $H^{\{123,124\}} = \{123, 124, 345\}$
- If F covers pairs then $H^{\{F\}} = F$

Lemma

$$\pi(H^F) = \pi_\lambda(F).$$

Remark: $\pi(K_4^3) = \pi_\lambda(K_4^3)$.

Turán number

Theorem(Keevash, Hefeez; Jiang, Peng, Wu; Peng, Wu;...)

Let $\mathcal{F}_1 = \{M_t^3 : t = 2, 3, \dots\} \cup \{M_t^4 : t = 4, 5, 6, 7, 8\} \cup \{P_3^3, P_4^3, P_2^4, P_2^5\}$ and let $F \in \mathcal{F}_1$. For sufficient large integer n,

$$ex(n, H^F) = t^r_{v(F)-1}(n).$$

Furthermore, the extremal hypergraph is $T^r_{v(F)-1}(n)$.

Theorem(Bene Watts, Norin, Yepremyan; Peng, Wu)

Let $\mathcal{F}_2 = \{M_2^r : r = 3, 4, 5, \dots\} \cup \{M_3^4\}$ and let $F \in \mathcal{F}$ with s edges. For sufficient large integer n,

$$ex(n, H^F) = e(blowup(\mathcal{S}^r_{s-1})).$$

Furthermore, the extremal hypergraph is $blowup(\mathcal{S}^r_{s-1})$.

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Thank you!