On a generalization of the bipartite graph $D(k, q)$

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Outline

Simple Graphs

The graphs we consider in this talk are simple, i.e. undirected, without loops and multiple edges.

For a graph G, its vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. The order of G is the number of vertices in $V(G)$.

The degree of a vertex $v \in G$ is the number of the vertices that are adjacent to it. A graph is said to be r-regular if the degree of every vertex is equal to r.

An automorphism of G means a bijection ϕ from $V(G)$ to itself such that $\{\phi(v), \phi(v')\}$ is an edge iff $\{v, v'\}$ is.

G is said to be edge-transitive if for any two edges $\{v_1, v_1'\}$, $\{v_2, v_2'\}$ there is an automorphism ϕ of G such that

$$\{\phi(v_1), \phi(v_1')\} = \{v_2, v_2'\}.$$

Simple Graphs

A sequence $v_1 v_2 \cdots v_n$ of vertices of G is called a path of length n if $\{v_i, v_{i+1}\} \in E(G)$ for $i = 1, 2, \ldots, n-1$ and $v_j \neq v_{j+2}$ for $j = 1, 2, \ldots, n-2$.

A path $v_1 v_2 \cdots v_n$ is called a cycle further if its length n is not smaller than 3 and $v_3 v_4 \cdots v_n v_1 v_2$ is still a path.

If G contains at least one cycle, then the girth of G, denoted by $g(G)$, is the length of a shortest cycle in G.

In the literature, graphs with *large girth* and a *high degree of symmetry* have been shown to be useful in different problems in *extremal graph theory, finite geometry, coding theory, cryptography, communication networks and quantum computations.*

Definition of $D(k, q)$

For prime power q and integer $k \geq 2$, Lazebnik et al. in 1995 proposed a bipartite graph, denoted by $D(k, q)$, which is *q-regular, edge-transitive* and of *large girth*.

The vertex sets $L(k)$ and $P(k)$ of $D(k, q)$ are two copies of $\mathbb{F}_q^k$ such that $(l_1, l_2, \ldots, l_k) \in L(k)$ and $(p_1, p_2, \ldots, p_k) \in P(k)$ are adjacent in $D(k, q)$ iff

$$l_2 + p_2 = p_1 l_1, \tag{1}$$

$$l_3 + p_3 = p_1 l_2, \tag{2}$$

and, for $4 \leq i \leq k$,

$$l_i + p_i = \begin{cases} p_{i-2} l_1, & \text{if } i \equiv 0 \text{ or } 1 \pmod 4, \\ p_1 l_{i-2}, & \text{if } i \equiv 2 \text{ or } 3 \pmod 4. \end{cases} \tag{3}$$

Girth of $D(k, q)$

In 1995 Lazebnik et al. showed some automorphisms of $D(k, q)$ and then proved $g(D(k, q)) \geq k + 4$.

Since the girth of a bipartite graph must be even, we have $g(D(k, q)) \geq k + 5$ for odd k. In 1995 Füredi et al. proved

$$g(D(k, q)) = k + 5$$

for odd k with $\frac{k+5}{2} | (q - 1)$ and conjectured further

Conjecture A. $g(D(k, q)) = k+5$ for all odd k and all $q \geq 4$.

This conjecture was proved to be valid:

- in 2014 when $k + 5 = 2p^s$ for some $s > 0$;
- in 2016 when $\frac{k+5}{2p^s}$ divides $q - 1$ for some $s \geq 0$;

where p is the characteristic of the field $\mathbb{F}_q$.

Connected Components of $D(k, q)$

The graph $D(k, q)$ was shown to be disconnected in general in 1996. The number of components of $D(k, q)$ was determined further for $q \geq 3$ in 2004.

The connectivity and the lower bound of the girth of $D(k, q)$ imply that the components of $D(k, q)$ provide the best-known asymptotic lower bound for the greatest number of edges in graphs of their order and girth. Indeed, at the present, the components of $D(k, q)$ provide the best general lower bound (for all but few exceptional values of t) on the Turán numbers $ex(n; \{C_3, \ldots, C_{2t+1}\})$ and $ex(n; \{C_{2t}\})$.

Wenger Graphs

For $n \geq 1$, Lazebnik and Viglione constructed in 2002 a bipartite graph $G_n(q)$ (indeed a generalization of the graph defined by Wenger in 1991) whose vertex sets are $L(n+1)$ and $P(n+1)$ such that two vertices $(l_1, l_2, \ldots, l_{n+1}) \in L(n+1)$ and $(p_1, p_2, \ldots, p_{n+1}) \in P(n+1)$ are adjacent in $G_n(q)$ if and only if

$$l_i + p_i = p_1 l_{i-1}, i = 2, 3, \ldots, n+1. \tag{4}$$

For $n \geq 3$ and $q \geq 3$, or $n = 2$ and q odd, the graph $G_n(q)$ is semi-symmetric(edge-transitive but not vertex-transitive), connected when $1 \leq n \leq q-1$ and disconnected when $n \geq q$, in which case it has q^{n-q+1} components, each isomorphic to $G_{q-1}(q)$.

Graphs Defined by Equation Systems

For $n \geq 2$ and commutative ring $\mathcal{R}$, Lazebnik and Woldar constructed in 2001 a bipartite graph $B\Gamma_n = B\Gamma(\mathcal{R}; f_2, \ldots, f_n)$ whose vertex sets L_n and P_n are two copies of $\mathcal{R}^n$ such that $(l_1, l_2, \ldots, l_n) \in L_n$ and $(p_1, p_2, \ldots, p_n) \in P_n$ are adjacent in $B\Gamma_n$ if and only if

$$l_i + p_i = f_i(p_1, l_1, p_2, l_2, \ldots, p_{i-1}, l_{i-1}), i = 2, 3, \ldots, n, \quad (5)$$

where $f_i : \mathcal{R}^{2i-2} \to \mathcal{R}$ are given functions.

Some general properties of $B\Gamma_n$ were exhibited by Lazebnik and Woldar in that paper.

When $\mathcal{R}$ is a finite field and the functions f_i are monomials of p_1 and l_1, the graph $B\Gamma_n$ is also called a monomial graph. For positive integers k, m, k', m', it is proved in 2005 that the monomial graphs $B\Gamma_2(\mathbb{F}_q; p_1^k l_1^m)$ and $B\Gamma_2(\mathbb{F}_q; p_1^{k'} l_1^{m'})$ are isomorphic if and only if $\{\gcd(k, q-1), \gcd(m, q-1)\} = \{\gcd(k', q-1), \gcd(m', q-1)\}$ as multi-sets.

If q is odd, it was proved in 2007 that any monomial graph $B\Gamma_3$ of girth ≥ 8 is isomorphic to a graph $B\Gamma_3(\mathbb{F}_q; p_1 l_1, p_1^k l_1^{2k})$ for k with $(k, q) = 1$. In particular, the integer k can be restricted to be 1 further if the odd prime power q is not greater than 10^{10} or of form $q = p^{2^a 3^b}$ for odd prime p. It was further conjectured that:

Conjecture B. For any odd prime power q, every monomial graph $B\Gamma_3$ of girth ≥ 8 is isomorphic to $B\Gamma_3(\mathbb{F}_q; p_1 l_1, p_1 l_1^2)$.

Two Conjectures on Permutation Polynomials

Let q be a power of an odd prime p and $1 \leq k \leq k-1$.

Conjection C. $X^k[(X+1)^k - x^k] \in \mathbb{F}_q[X]$ is a permutation polynomial if and only if k is a power of p.

Conjection D. $[(X+1)^{2k} - 1]X^{q-1-k} - 2X^{q-1} \in \mathbb{F}_q[X]$ is a permutation polynomial if and only if k is a power of p.

It was proved in 2007 that Conjecture B is true if Conjectures C and D are true.

In 2019, Xiangdong Hou et.al proved that Conjecture B is true though the Conjectures C and D are still open.

Condition for the Index Sets

In this talk, T denotes a set of binary sequences satisfying

C1: T contains the null sequence η and the sequences obtained from those in $T \setminus \{\eta\}$ by deleting the last bits.

We note that T corresponds indeed a binary tree.

Let $L(T)$ and $R(T)$ be two copies of $\mathbb{F}_q^{|T|+1}$, where $|T|$ is the size of T.

We will denote the vectors in $L(T)$ by $[l]$ (or in $R(T)$ by $\langle r \rangle$, respectively). The entries of vectors in $L(T) \cup R(T)$ are indexed by the elements in $T \cup \{*\}$, where $*$ is a symbol not in T and used to index the colors of the vectors.

Definition of the Generalized Graphs

Let $\Gamma(T, q)$ be the bipartite graph with vertex sets $L(T)$, $R(T)$ such that $[l] \in L(T)$ and $\langle r \rangle \in R(T)$ are adjacent in $\Gamma(T, q)$ if and only if

$$l_\eta + r_\eta = l_* r_*, \tag{6}$$

and

$$l_{\alpha 0} + r_{\alpha 0} = r_* l_\alpha, \text{ for } \alpha 0 \in T, \tag{7}$$

$$l_{\beta 1} + r_{\beta 1} = l_* r_\beta, \text{ for } \beta 1 \in T. \tag{8}$$

If we define $*0\sigma = *1\sigma = \sigma$ for any $\sigma \in \{0, 1\}^*$, the equation (6) can also be included in either (7) or (8).

Generalization of $D(k, q)$ and $G_n(q)$

Let $[0]_x$ denote the vector $[l]$ satisfying $l_* = x$ and $l_\alpha = 0$ for $\alpha \in T$, and $\langle 0 \rangle_x$ denote the vector $\langle r \rangle$ satisfying $r_* = x$ and $r_\alpha = 0$ for $\alpha \in T$.

Clearly, for any T the bipartite graph $\Gamma(T, q)$ is q-regular and contains the edge $([0]_0, \langle 0 \rangle_0)$.

For $k \geq 2$, let U_k denote the set consisting of the first $k-1$ elements in the following set

$$U = \{\eta, 0, 1, 01, 10, 010, 101, 0101, 1010, \ldots\}. \qquad (9)$$

Then, $\Gamma(U_k, q)$ is equivalent to $D(k, q)$.

For positive integer n, let W_n denote the set of all-zero sequences of length less than n. Then, $\Gamma(W_n, q)$ is equivalent to the Wenger graph $G_n(q)$.

Automorphisms of $\Gamma(T, q)$

For any binary sequence α, let $|\alpha|$ and $w(\alpha)$ denote its length and the number of its nonzero bits, respectively.

For $x, y \in \mathbb{F}_q$, let $\lambda_{x,y}$ denote the map from $L(T) \cup R(T)$ to itself such that, for $[l] \in L(T)$ and $\langle r \rangle \in R(T)$,

$$(\lambda_{x,y}([l]))_* = xl_*, (\lambda_{x,y}(\langle r \rangle))_* = yr_*,$$

and

$$(\lambda_{x,y}([l]))_\alpha = x^{w(\alpha)+1} y^{|\alpha|-w(\alpha)+1} l_\alpha,$$
$$(\lambda_{x,y}(\langle r \rangle))_\alpha = x^{w(\alpha)+1} y^{|\alpha|-w(\alpha)+1} r_\alpha.$$

Theorem

For any $x, y \in \mathbb{F}_q^*$, $\lambda_{x,y}$ is an automorphism of $\Gamma(T, q)$.

Automorphisms of $\Gamma(T, q)$

For $\alpha \in T$, let $S_T(\alpha)$ denote the set of sequences β with

$$\{\alpha 01\beta, \alpha 10\beta\} \cap T \neq \emptyset.$$

Let $\mathbb{S}(T) = \cup_{\alpha \in T} S_T(\alpha)$.

Theorem

For any $x \in \mathbb{F}_q$ and $\alpha \in T$ with $S_T(\alpha) \subset T$, there is an automorphism $\theta_{x,\alpha}$ of $\Gamma(T, q)$ such that, for any $[l] \in L(T)$ and $\langle r \rangle \in R(T)$,

$$(\theta_{x,\alpha}([l]))_\alpha = l_\alpha + x, \ (\theta_{x,\alpha}(\langle r \rangle))_\alpha = r_\alpha - x,$$

and

$$(\theta_{x,\alpha}([l]))_\gamma = l_\gamma, \ (\theta_{x,\alpha}(\langle r \rangle))_\gamma = r_\gamma,$$

for any $\gamma \notin \{\alpha\beta : \beta \in \{0, 1\}^*\}$.

Automorphisms of $\Gamma(T, q)$

For $\sigma \in \{0,1\}^*$, let $\sigma^0 = \eta$ and $\sigma^i = \sigma^{i-1}\sigma$ for $i > 0$.

Theorem

If $\mathbb{S}(T) \subset T$, then for any edge $([l], \langle r \rangle)$ there are automorphisms θ_0, θ_1 of $\Gamma(T, q)$ such that

1. $\theta_0([l]) = [0]_{l_*}$, $(\theta_0(\langle r \rangle))_* = r_*$ *and*

$$(\theta_0(\langle r \rangle))_{1^i} = l_*^{i+1} r_*, \ \textit{if } 1^i \in T, i \geq 0,$$
$$(\theta_0(\langle r \rangle))_\alpha = 0 \textit{ for all } \alpha \in T \setminus \{1^i : i \geq 0\},$$

2. $\theta_1(\langle r \rangle) = \langle 0 \rangle_{r_*}$, $(\theta_1([l]))_* = l_*$ *and*

$$(\theta_1([l]))_{0^i} = r_*^{i+1} l_*, \ \textit{if } 0^i \in T, i \geq 0,$$
$$(\theta_1([l]))_\alpha = 0 \textit{ for all } \alpha \in T \setminus \{0^i : i \geq 0\}.$$

Automorphisms of $\Gamma(T, q)$

For $a \in \{0, 1\}$, let $H_a(T)$ denote the set of sequences obtained from the sequences $\alpha \in T \setminus \{\eta\}$ by deleting a a either from the leftmost or from two consecutive a's.

Theorem

1. *If $H_0(T) \subset T$, then, for any $x \in \mathbb{F}_q$, there is an automorphism ϕ of $\Gamma(T, q)$ such that*

$$(\phi([l]))_* = l_*, \text{ for } [l] \in L(T),$$
$$(\phi(\langle r \rangle))_* = r_* + x, \text{ for } \langle r \rangle \in R(T).$$

2. *If $H_1(T) \subset T$, then, for any $x \in \mathbb{F}_q$, there is an automorphism ψ of $\Gamma(T, q)$ such that*

$$(\psi([l]))_* = l_* + x, \text{ for } [l] \in L(T),$$
$$(\psi(\langle r \rangle))_* = r_*, \text{ for } \langle r \rangle \in R(T).$$

Automorphisms of $\Gamma(T, q)$

Theorem

1. *If* $H_0(T) \cup \mathbb{S}(T) \subset T$, *then for any* $\langle r \rangle \in R(T)$ *there is an automorphism* π *of* $\Gamma(T, q)$ *such that* $\pi(\langle r \rangle) = \langle 0 \rangle_0$.
2. *If* $H_1(T) \cup \mathbb{S}(T) \subset T$, *then for any* $[l] \in L(T)$ *there is an automorphism* π *of* $\Gamma(T, q)$ *such that* $\pi([l]) = [0]_0$.

Theorem

If $H_0(T) \cup H_1(T) \subset T$, *then the bipartite graph* $\Gamma(T, q)$ *is edge-transitive, or equivalently, for any edge* $([l], \langle r \rangle) \in E(\Gamma(T, q))$ *there is an automorphism* π *of* $\Gamma(T, q)$ *such that* $\pi([l]) = [0]_0$ *and* $\pi(\langle r \rangle) = \langle 0 \rangle_0$.

Corollary

The bipartite graphs $D(k, q)$ *and* $G_k(q)$ *are edge-transitive.*

Example 1

For example, let T_1 denote the following set

$$\{\eta, 1, 0, 10, 0^2, 01, 101, 0^21, 010, (10)^2, 0^210, (01)^2, (10)^21, 0(01)^2, (01)^20, (10)^3, 0^2(10)^2, (01)^3, 1(01)^3, 0(01)^3\}$$

Then, the bipartite graph $\Gamma(T_1, q)$ is edge-transitive.

If the sequences in T_1 are mapped into $\{1, 2, \ldots, 20\}$ in order and the symbol $*$ is mapped to 0, then $[l]$ and $\langle r\rangle$ are adjacent in $\Gamma(T_1, q)$ if and only if

$$l_k + r_k = l_0 r_{k-1},\ k = 1, 2,$$
$$l_j + r_j = r_0 l_{j-2},\ j = 3, 4, 5,$$

and, for $6 \leq i \leq 20$,

$$l_i + r_i = \begin{cases} l_0 r_{i-3}, & \text{if } i \equiv 0 \text{ or } 1 \text{ or } 2 (\text{mod} 6), \\ r_0 l_{i-3}, & \text{if } i \equiv 3 \text{ or } 4 \text{ or } 5 (\text{mod} 6). \end{cases}$$

Invariants for Components of $\Gamma(T, q)$

For any $x \in \mathbb{F}_q$, let x^0 be the multiplicative unit of $\mathbb{F}_q$.

Lemma

For any $([l], \langle r\rangle) \in E(\Gamma(T, q))$, $\alpha, \beta \in \{0,1\}^* \cup \{*\}$ *and* $s \geq 0$,

1. *If* $\{\alpha 10^s, \beta 10^s\} \subset T$, *then*

$$l_{\alpha 10^s} r_\beta - r_\alpha l_{\beta 10^s} = \sum_{t=0}^{s} r_*^{s-t} (r_\alpha r_{\beta 10^t} - r_{\alpha 10^t} r_\beta).$$

2. *If* $\{\alpha 01^s, \beta 01^s\} \subset T$, *then*

$$r_{\alpha 01^s} l_\beta - l_\alpha r_{\beta 01^s} = \sum_{t=0}^{s} l_*^{s-t} (l_\alpha l_{\beta 01^t} - l_{\alpha 01^t} l_\beta).$$

Invariants for Components of $\Gamma(T, q)$

Theorem

Suppose $\alpha \in \{0,1\}^* \cup \{*\}$.

1. *If* $\alpha 0^q$ *is a sequence in* T, *then*

$$l_{\alpha 0^q} - l_{\alpha 0} = r_{\alpha 0} - (r_{\alpha 0^q} + r_* r_{\alpha 0^{q-1}} + \cdots + r_*^{q-1} r_{\alpha 0})$$

is an invariant of $\Gamma(T, q)$.

2. *If* $\alpha 1^q$ *is a sequence in* T, *then*

$$r_{\alpha 1^q} - r_{\alpha 1} = l_{\alpha 1} - (l_{\alpha 1^q} + l_* l_{\alpha 1^{q-1}} + \cdots + l_*^{q-1} l_{\alpha 1})$$

is an invariant of $\Gamma(T, q)$.

Invariants for Components of $\Gamma(T, q)$

For any sequence $\boldsymbol{s} = (s_1, s_2, \ldots)$ of nonnegative integers, let $\mu_0(\boldsymbol{s}) = *$ and, for $i \geq 1$,

$$\mu_i(\boldsymbol{s}) = \begin{cases} \mu_{i-1}(\boldsymbol{s})10^{s_i}, & \text{if } i \text{ is odd,} \\ \mu_{i-1}(\boldsymbol{s})01^{s_i}, & \text{if } i \text{ is even.} \end{cases}$$

Theorem

If $\alpha = \mu(s_1, \ldots, s_{2n+1})$ is not equal to $\hat{\alpha} = \mu(s_{2n+1}, \ldots, s_1)$, then for any edge $([l], \langle r \rangle)$ of $\Gamma(T, q)$,

$$l_\alpha - l_{\hat{\alpha}} + \sum_{i=1}^{n} \sum_{t=0}^{s_{2i}} l_*^{s_{2i}-t}(l_{\mu'_{2i}} l_{\mu_{2i-1}01^t} - l_{\mu'_{2i}01^t} l_{\mu_{2i-1}}),$$

$$= r_{\hat{\alpha}} - r_\alpha + \sum_{i=0}^{n} \sum_{t=0}^{s_{2i+1}} r_*^{s_{2i+1}-t}(r_{\mu'_{2i+1}10^t} r_{\mu_{2i}} - r_{\mu'_{2i+1}} r_{\mu_{2i}10^t}),$$

where $\mu_i = \mu_i(s_1, \ldots, s_{2n+1})$ and $\mu'_i = \mu_{2n+1-i}(s_{2n+1}, \ldots, s_1)$.

Invariants for Components of $\Gamma(T, q)$

For any sequence $\boldsymbol{s} = (s_1, s_2, \ldots)$ of nonnegative integers, let $\nu_0(\boldsymbol{s}) = *$ and, for $i \geq 1$,

$$\nu_i(\boldsymbol{s}) = \begin{cases} \nu_{i-1}(\boldsymbol{s})01^{s_i}, & \text{if } i \text{ is odd,} \\ \nu_{i-1}(\boldsymbol{s})10^{s_i}, & \text{if } i \text{ is even.} \end{cases}$$

Theorem

If $\alpha = \nu(s_1, \ldots, s_{2n+1})$ is not equal to $\hat{\alpha} = \nu(s_{2n+1}, \ldots, s_1)$, then for any edge $([l], \langle r \rangle)$ of $\Gamma(T, q)$,

$$l_\alpha - l_{\hat{\alpha}} + \sum_{i=0}^{n} \sum_{t=0}^{s_{2i+1}} l_*^{s_{2i+1}-t} (l_{\nu'_{2i+1}} l_{\nu_{2i}01^t} - l_{\nu'_{2i+1}01^t} l_{\nu_{2i}})$$
$$= r_{\hat{\alpha}} - r_\alpha + \sum_{i=1}^{n} \sum_{t=0}^{s_{2i}} r_*^{s_{2i}-t} (r_{\nu'_{2i}10^t} r_{\nu_{2i-1}} - r_{\nu'_{2i}} r_{\nu_{2i-1}10^t}),$$

where $\nu_i = \nu_i(s_1, \ldots, s_{2n+1})$ and $\nu'_i = \nu_{2n+1-i}(s_{2n+1}, \ldots, s_1)$.

Invariants for Components of $\Gamma(T, q)$

Theorem

If $\alpha = \mu(s_1, \ldots, s_{2n})$ is not equal to $\hat{\alpha} = \nu(s_{2n}, \ldots, s_1)$, then for any two adjacent vertices $[l], \langle r \rangle$ of $\Gamma(T, q)$,

$$l_\alpha - l_{\hat{\alpha}} + \sum_{i=1}^{n} \sum_{t=0}^{s_{2i}} l_*^{s_{2i}-t}(l_{\nu''_{2i}} l_{\mu_{2i-1}01^t} - l_{\nu''_{2i}01^t} l_{\mu_{2i-1}})$$

$$= r_{\hat{\alpha}} - r_\alpha + \sum_{i=0}^{n-1} \sum_{t=0}^{s_{2i+1}} r_*^{s_{2i+1}-t}(r_{\nu''_{2i+1}10^t} r_{\mu_{2i}} - r_{\nu''_{2i+1}} r_{\mu_{2i}10^t}),$$

where $\mu_i = \mu_i(s_1, \ldots, s_{2n})$ and $\nu''_i = \mu_{2n-i}(s_{2n}, \ldots, s_1)$.

Lower Bounds of the Number of Components

For any binary sequence $\alpha = a_1a_2\cdots a_n$, $a_i \in \{0, 1\}$, let $\hat{\alpha} = a_na_{n-1}\cdots a_1$.

Theorem

$\Gamma(T, q)$ has at least q^h connected components, where h is the number of pairs $\alpha, \hat{\alpha} \in T$ with $\alpha \neq \hat{\alpha}$.

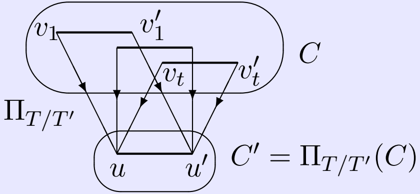
Projection from $\Gamma(T,q)$ to $\Gamma(T',q)$

Hereafter, we suppose that T' is a subtree of T with $\eta \in T'$.

Let $\Pi_{T/T'}$ denote the projection from $\Gamma(T,q)$ to $\Gamma(T',q)$ defined naturally.

Theorem

For any component C of $\Gamma(T,q)$, $\Pi_{T/T'}(C)$ is a component of $\Gamma(T',q)$ and $\Pi_{T/T'}$ is a t-to-1 graph homomorphism from C to $\Pi_{T/T'}(C)$ for some t with $1 \le t \le q^{|T|-|T'|}$.



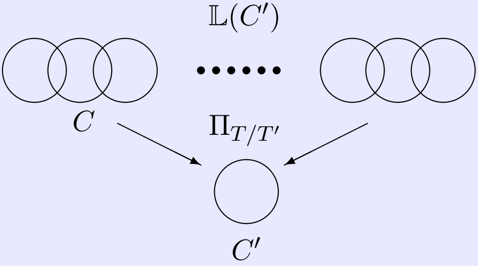
where $\{v_1, \ldots, v_t\} = \{v \in V(C) : \Pi_{T/T'}(v) = u\}$.

Lifting to a Component of $\Gamma(T, q)$

Assume $T' = T \setminus \{\alpha\}$ for some leaf node α in the tree T. Let C' be a component of $\Gamma(T', q)$ and $\mathbb{L}(C')$ the set of components C of $\Gamma(T, q)$ with $C' = \Pi_{T/T'}(C)$. For $C \in \mathbb{L}(C')$ and vertex $u \in V(C')$, let

$$\rho(u, C) = \{v_\alpha | v \in V(C), \Pi_{T/T'}(v) = u\}.$$

Then, $\rho(u, C)$ is a coset of some additive subgroup G of $\mathbb{F}_q$.



Structure of the Lifting Sets

Theorem

Suppose $T' = T \setminus \{\alpha\}$ for some leaf node α in the tree T. Let C' be a component of $\Gamma(T', q)$ and $C \in \mathbb{L}(C')$. Then, $n = |\mathbb{L}(C')|$ divides q and there are maps $f : V(C') \to \mathbb{F}_q$, $g : \mathbb{L}(C') \to \mathbb{F}_q$ and an additive subgroup $G = G(C')$ of $\mathbb{F}_q$ of order $t = q/n$ such that, for $C \in \mathbb{L}(C')$,

$$\rho(u, C) = \begin{cases} f(u) + g(C) + G, & \text{if } u \in V(C') \cap L(T'), \\ f(u) - g(C) + G, & \text{if } u \in V(C') \cap R(T'), \end{cases}$$

where $\{g(C) | C \in \mathbb{L}(C')\}$ is a representive set of cosets of G in $\mathbb{F}_q$, namely, $\{g(C) + G\}_{C \in \mathbb{L}(C')}$ are distinct cosets of G in $\mathbb{F}_q$ with $\bigcup_{C \in \mathbb{L}(C')} (g(C) + G) = \mathbb{F}_q$.

Projections and Lifts

Theorem

Suppose $\mathbb{S}(T) \subset T$ and $\mathbb{S}(T') \subset T'$.

1. *There is an integer t with $1 \leq t \leq q^{|T|-|T'|}$ such that, for any component C of $\Gamma(T, q)$, $\Pi_{T/T'}$ is a t-to-1 graph homomorphism from C to $\Pi_{T/T'}(C)$.*
2. *If $T' = T \setminus \{\alpha\}$ for some leaf node α of T, there is an additive subgroup G of $\mathbb{F}_q$ such that, for any component C of $\Gamma(T, q)$ and vertex u of $\Pi_{T/T'}(C)$, the set $\rho(u, C)$ is a coset of G.*

Corollary

If $\mathbb{S}(T) \subset T$, all components of $\Gamma(T, q)$ are of the same size.

Lower Bounds for the Girth of $\Gamma(T, q)$

For $a \in \{0, 1\}$, let M_a denote the set of the sequences $\beta \in U$ that are lead by a, namely,

$$M_a = \begin{cases} \{0, 01, 010, 0101, \ldots\}, & \text{if } a = 0, \\ \{1, 10, 101, 1010, \ldots\}, & \text{if } a = 1. \end{cases}$$

Lemma

Let β be a sequence in $T \cap U$.

1. *If $\beta \in M_0$, then $\Gamma(T, q)$ has no cycle of length $2(|\beta| + 1)$ containing a vertex of form $[0]_x$.*
2. *If $\beta \in M_1$, then $\Gamma(T, q)$ has no cycle of length $2(|\beta| + 1)$ containing a vertex of form $\langle 0 \rangle_x$.*

Theorem

Assume that $\mathbb{S}(T) \subset T$. If $\beta \in T \cap U$, then the girth of $\Gamma(T, q)$ is at least $2(|\beta| + 2)$.

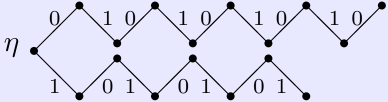
Example 2

For example, for $n > 0$ let

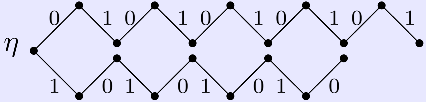
$$\mathcal{X}_n = \begin{cases} U_{4k-1} \cup \{(01)^{k-1}0, (01)^k 0\}, & \text{if } n = 2k-1, \\ U_{4k+1} \cup \{(01)^k, (01)^{k+1}\}, & \text{if } n = 2k. \end{cases}$$

From $\mathbb{S}(\mathcal{X}_n) \subset \mathcal{X}_n$, we see that the girth of $\Gamma(\mathcal{X}_n, q)$ is at least $2((n+2)+2) = 2n+8$. Note that this lower bound is equal to the best known one achieved by $D(2n+3, q) \cong \Gamma(U_{2n+3}, q)$.

$\mathcal{X}_{2k-1}$:



$\mathcal{X}_{2k}$:



Lower Bounds for the Girth of $\Gamma(T, q)$

Lemma

Let s, t be nonnegative integers with $\gcd(s-t, q-1) = 1$*.*

1. *If $\beta \in M_0 \cup \{\eta\}$ and $\{0^s\beta, 0^t\beta\} \subset T$, then $\Gamma(T, q)$ has no cycle of length $2(|\beta| + 2)$ containing a vertex of form $[0]_x$.*
2. *If $\beta \in M_1 \cup \{\eta\}$ and $\{1^s\beta, 1^t\beta\} \subset T$, then $\Gamma(T, q)$ has no cycle of length $2(|\beta| + 2)$ containing a vertex of form $\langle 0 \rangle_x$.*

Theorem

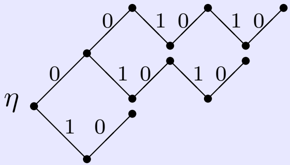
Suppose $\mathbb{S}(T) \subset T$, $a \in \{0, 1\}$ and $\beta \in M_a \cup \{\eta\}$. If there are nonnegative integers s, t with $\gcd(s-t, q-1) = 1$ *such that $\{a^s\beta, a^t\beta\} \subset T$, then the girth of $\Gamma(T, q)$ is at least $2(|\beta| + 3)$.*

Example 3

Let T_3 denote the following set

$$\{\eta, 0, 1, 01, 10, 010, 0101, 01010, 0^2, 0^21, 0^210, 0^2101, 0^21010\}.$$

From $\mathbb{S}(T_3) \cup \{01010, 001010\} \subset T_3$, the girth of $\Gamma(T_3, q)$ is at least $2(5+3) = 16$.



Lower Bounds for the Girth of $\Gamma(T, q)$

Lemma

1. *If* $\beta \in M_0 \cup \{\eta\}$ *and* $1^m\beta \in T$ *for some* $m > 0$, *then* $\Gamma(T, q)$ *has no cycle of length* $2(|\beta| + 3)$ *containing the vertex* $[0]_0$.
2. *If* $\beta \in M_1 \cup \{\eta\}$ *and* $0^m\beta \in T$ *for some* $m > 0$, *then* $\Gamma(T, q)$ *has no cycle of length* $2(|\beta| + 3)$ *containing the vertex* $\langle 0 \rangle_0$.

Theorem

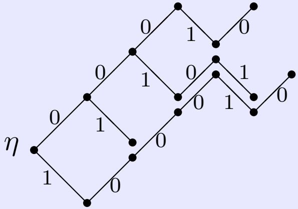
Suppose $a \in \{0, 1\}$ *and* $\mathbb{S}(T) \cup H_a(T) \subset T$. *If there are positive integers* $t, n_1, n_2, \ldots, n_t, m$ *and sequences* $\gamma, \beta \in M_a \cup \{\eta\}$ *with* $|\beta| = |\gamma| + 2t - 1$ *such that* $\{a\bar{a}^{n_1} a\bar{a}^{n_2} \cdots a\bar{a}^{n_t}\gamma, \bar{a}^m\beta\} \subset T$, *where* $\bar{a}$ *is the symbol in* $\{0, 1\}$ *other than* a, *then the girth of* $\Gamma(T, q)$ *is at least* $2(|\beta| + 4)$.

Example 4

Let T_4 denote the set of the following sequences

$$10^3 10, 10^3 1, 10^3, 10^2, 10, 1;$$
$$0^2 101, 0^2 10, 0^2 1, 0^2, 0; 01; 0^3 10, 0^3 1, 0^3; \eta.$$

From $\mathbb{S}(T_4) \cup H_1(T_4) \cup \{10^3 10, 0^2 101\} \subset T_4$, the girth of $\Gamma(T_4, q)$ is at least $2(3+4) = 14$.



Lower Bounds for the Girth

Theorem

Assume that $H_0(T) \cup H_1(T) \subset T$. *If* $\alpha \in T \cap U$, *then the girth of* $\Gamma(T, q)$ *is at least* $2(|\alpha| + 3)$.

Corollary

For $k \geq 2$, *the girth of* $D(k, q)$ *is at least* $k + 4$.

Conclusion

In this talk, we deal with a generalization $\Gamma(T, q)$ of the bipartite graph $D(k, q)$, by indexing the entries of vertex vectors with the nodes in a binary tree T.

1. Sufficient conditions for $\Gamma(T, q)$ to admit a variety of automorphisms are proposed. A sufficient condition for $\Gamma(T, q)$ to be edge-transitive is shown.
2. For $\Gamma(T, q)$, we show some invariants over the connected components. A lower bound for the number of its components is given.
3. Projections and lifts of $\Gamma(T, q)$ are investigated.
4. A few lower bounds for the girth of $\Gamma(T, q)$ are deduced. New families of graphs with large girth in the sense of Biggs can be obtained from $\Gamma(T, q)$, such as $\Gamma(\mathcal{X}_n, q)$.

Thank You for Your Attention!