Transversal Packing designs

Lijun Ji

Department of Mathematics
Soochow University

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Transversal Packing Designs

An H(m, g, k, t) design is a triple $(X, \mathcal{T}, \mathcal{B})$, where

(1) X is a set of mg points,
(2) $\mathcal{T}$ is a partition of X into m disjoint sets of size g (called groups), and
(3) $\mathcal{B}$ is a set of k-element transverses of $\mathcal{T}$, such that each t-element transverse of $\mathcal{T}$ is contained in at most one of them.

If each t-element transverse is contained in exactly one block, then it is called an $H(m, g, k, t)$ design (or a group divisible t-design of type g^m).

An H$(m, 1, k, t)$ design is usually called a Steiner system and denoted by S(t, k, m), the group set $\mathcal{T}$ is omitted, i.e., a Steiner system is a pair $(X, \mathcal{B})$, where X is a set of m points and $\mathcal{B}$ is a set of k-subsets of X such that each t-subset of X is contained in exactly one of $\mathcal{B}$

An S$(2, k, m)$ is called a balanced incomplete block design.

Example: TD$(k-1,k,g)$

An H(k,g,k,t) design is also called a transversal design and denoted by TD(t,k,g).

TD$(k-1,k,g)$

- Point set: $I_k \times \mathbb{Z}_g$, $I_k = \{1,2,\ldots,k\}$;
- Groups: $\{i\} \times \mathbb{Z}_g$, $i \in I_k$;
- Blocks: $\{(1,x_1),(2,x_2),\ldots,(k,x_k)\}$,
  $$x_1,\ldots,x_k \in \mathbb{Z}_g,\ x_1+\cdots+x_k \equiv 0 \pmod g.$$

Equivalence

A TD$(2, n+1, n)$ exists.

$\iff$ There are $n-1$ mutually Latin squares of order n.

$\iff$ An S$(2, n+1, n^2+n+1)$ exists.

$\iff$ A projective plane of order n exists.

$\iff$ An affine plane of order n exists.

Necessary Conditions

H(m, g, k, t) design $\Longrightarrow$ H$(m-1, g, k-1, t-1)$

Theorem

If there is an H(m, g, k, t) design, then

$$\binom{m-i}{t-i} g^{t-i} \equiv 0 \pmod{\binom{k-i}{t-i}},$$

for $0 \leq i < t$.

Construction Methods:

- Difference family $\Longrightarrow$ S$(2, k, n)$
- Relative difference family $\Longrightarrow$ H$(n, g, k, 2)$
- Difference matrix $\Longrightarrow$ TD$(2, k, n)$
- Wilson's fundamental construction

Difference Matrix

Let G be an abelian group of order n. A difference matrix, or an $(n, k; 1)$-DM is a $k \times n$ array (a_{ij}) $(1 \leq i \leq k$, $1 \leq j \leq n)$ with entries from G, such that, for any two distinct rows l and h of D $(1 \leq l < h \leq k)$, the difference list

$$\Delta_{lh} = \{d_{h1} - d_{l1}, d_{h2} - d_{l2}, \ldots, d_{hn} - d_{ln}\}$$

contains every element of G exactly once.

H$(m, g, k, 2)$ designs

- For $k \in \{3, 4\}$, the existence of an H$(m, g, k, 2)$ has been determined.
- The existence of an H$(m, g, 5, 2)$ has been almost determined with a few possible exceptions.
- There is a TD$(2, q+1, q)$ for any prime power q.

C. J. Colbourn and J. H. Dinitz, The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 2007.

- H$(m, 1, 4, 3)$ designs, Hanani 1960 [1]
- H$(m, g, 4, 3)$ designs, $m \neq 5$, Mills 1990 [2]
- H$(5, g, 4, 3)$ designs, Ji 2019 [3]

Theorem

An H$(m, g, 4, 3)$ design exists if and only if $m \geq 4$, $mg \equiv 0 \pmod 2$, $g(m-1)(m-2) \equiv 0 \pmod 3$, and $(m, g) \neq (5, 2)$.

[1] H. Hanani, On quadruple systems, Canad. J. Math. 12 (1960), 145-157.

[2] W. H. Mills, On the existence of H designs, Congr. Numer. 79 (1990), 129-141.

[3] L. Ji, Existence of H designs, J. Combin. Des. 27 (2019), 75-81

A *candelabra* t*-system* of order $mg+s$ and block sizes from K is a quadruple $(X, S, \Gamma, \mathcal{A})$, denoted by $CS(t, K, mg+s)$ of type $(g^m : s)$, where:

(1) X is a set of $mg+s$ elements (called *points*);
(2) S is an s-subset of X (called a *stem*);
(3) $\Gamma = \{G_1, \ldots, G_m\}$ is a partition $X \backslash S$ into m groups of size g;
(4) $\mathcal{A}$ is a family of subsets of X, each of cardinality from K (called *blocks*);
(5) every t-subset T of X with $|T \cap (S \cup G_i)| < t$ for all i, is contained in a unique block and no t-subset of $S \cup G_i$ for all i occurs.

Fundamental construction for 3-CSs

A $CS(3,4,mg+s)$ of type $(g^n : s)$ is called a candelabra quadruple system and denoted by $CQS(g^n : s)$.

Theorem (Hartman 1994)

Suppose that there is an $S(3,K,v+1)$. If there is an H$(k,g,4,3)$ design and a $CQS(g^{k-1} : s)$ for $k \in K$, there is a $CQS(g^v : s)$. [a]

[a] A. Hartman, The fundamental construction for 3-designs, Discrete Math. 124 (1994), 107-132.

Existence of SQSs (Hanani 1960)

- An $S(3, \{4, 6\}, v)$ exists $\Longleftrightarrow v \equiv 0 \pmod 2$ (Hanani 1963).
- There is an $S(3, \{4, 6\}, v)$ whose blocks of size 6 form a partition of the point set (called a G$(v/6, 6, 4, 3)$ design) if $v \equiv 0 \pmod 6$ (Mills 1974).
- An SQS(v) exists if $v \equiv 4 \pmod 6$.
- An SQS(v) exists if $v \equiv 8 \pmod{12}$.
- An SQS(v) exists if $v \equiv 2 \pmod{12}$.

An $H((n^m : s), g, k, t)$ *frame* is an ordered four-tuple $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ where

(1) X is a set of $mng + sg$ *points*;

(2) $\mathcal{G} = \{G_1, \ldots, G_{mn+s}\}$ is an equipartition of X into $mn + s$ *groups*;

(3) $\mathcal{F} = \{F_0, F_1, \ldots, F_m\}$ is a family of subsets of $\mathcal{G}$ called *holes* such that $|F_1| = \cdots = |F_m| = n + s$, $|F_0| = s$ and $F_i \cap F_j = F_0$, $1 \leq i < j \leq m$;

(4) $\mathcal{B}$ is a set of k-transverses (called *blocks*) of $\mathcal{G}$ with the property that blocks contain exactly each t-transverses of $\mathcal{G}$ which is *not* a t-transverse of some hole $F_i \in \mathcal{F}$ once, no other t-transverses.

Existence of H designs with $m \neq 5$ (Mills 1990)

- SQS$(m) \Longrightarrow$ H$(m, g, 4, 3)$ for $m \equiv 2, 4 \pmod 6$ and any positive integer g.
- S$(3, \{4, 6\}, m) \Longrightarrow$ H$(m, g, 4, 3)$ for $m \equiv 0 \pmod 6$ and $g \equiv 0 \pmod 3$.
- G-designs and SQSs $\Longrightarrow$ H$(m, g, 4, 3)$ for $m \equiv 1, 5 \pmod 6$ and $g \equiv 0 \pmod 2$.
- H frame $\Longrightarrow$ H$(m, g, 4, 3)$ for $m \equiv 3 \pmod 6$ and $g \equiv 0 \pmod 6$.

Incomplete H designs IH$(m,(g,g'),k,t)$

An IH$(m,(g,g'),k,t)$ (H(m,g,k,t)-H(m,g',k,t)) is a partial H(m,g,k,t) design $(X,\mathcal{G},\mathcal{B})$ with a hole Y of size mg' satisfying the following conditions:

(1) $|Y \cap G| = g'$ for any $G \in \mathcal{G}$,

(2) each t-transverse T of $\mathcal{G}$ is contained in exactly one block of $\mathcal{B}$ if $T \not\subset Y$, and

(3) any t-transverse T of $\mathcal{G}$ with $T \subset Y$ is not contained in any block of $\mathcal{B}$.

A generalized lattice (t,ℓ)-design

A GLD$(m, n\times g, k, (t,\ell))$ is partial H(m, ng, k, t) design $(X, \mathcal{G}, \mathcal{B})$ with a hole set $\mathcal{H}$ such that

(1) $\mathcal{H}$ is a partition of X into n subsets of size mg,
(2) $|G\cap H| = g$ for $G\in\mathcal{G}$, $H\in\mathcal{H}$,
(3) $|B\cap H| \leq \ell$ for $B\in\mathcal{B}$, $H\in\mathcal{H}$, and
(4) each t-subset of X, which meets each $G\in\mathcal{G}$ in at most one point and meets each $H\in\mathcal{H}$ in at most ℓ points, is contained in exactly one block.

When $g = 1$ and $\ell = 1$, it is called a lattice t-design and shortly denoted by LD(m, n, k, t).

In 2002, Moh$\acute{a}$csy and Ray-Chaudhuri pointed out an equivalence between ordered designs and Lattice designs.

Teirlinck 1990

There is an LD$(m, 4, 4, 3)$ for any integer $m \geq 4$ with $m \neq 7$.

Lemma

For any integer $g \equiv 2, 10 \pmod{12}$ with $n \geq 10$, there is an $\mathrm{H}(5, g, 4, 3)$ design.

Proof: Write $g = 2g'$. Start with an $\mathrm{SQS}(g' + 1)$ or a $\mathrm{G}(\frac{g'+1}{6}, 6, 4, 3)$. Input a $\mathrm{GLD}(5, 3 \times 2, 4, (3, 2))$, a $\mathrm{GLD}(5, k \times 2, 4, (3, 1))$ and an $\mathrm{IH}(5, (k, 2), 4, 3)$, $k \in \{4, 6\}$. The required design is obtained.

Backgroud of Transversal Designs

- K. A. Bush, Orthogonal arrays of index unity, *Ann. Math. Stat.* 23 (1952) 426-434.
- A. S. Hedayat, N. J. A. Sloane and J. Stufken, Orthogonal Arrays Theory and Applications, Springer-Verlag, New York, 1999.

Example: TD$(t, q+1, q)$, q a prime power

TD$(t, q+1, q)$ (Bush 1952)

- Point set: $I_{q+1} \times GF(q)$, $GF(q) = \{\alpha_1, \ldots, \alpha_q\}$;
- Groups: $\{i\} \times GF(q)$, $i \in I_{q+1}$;
- Blocks:
  $\{(1, f(\alpha_1)), \ldots, (q, f(\alpha_q)), (q+1, [x^{t-1}]f(x))\}$,
  $f(x) \in GF(q)[x]$, $deg(f(x)) < t$.

Equivalence

A TD(t,k,n) exists.

$\Longleftrightarrow$ An OA(t,k,n) exists

$\Longleftrightarrow$ There is an n-ary maximal distance separable code of length k and Hamming distance $k-t+1$.

An orthogonal array OA(t,k,n) is an n^t by k array with entries from a symbol set X of size n such that each of its $n^t \times t$ subarrays contains every t-tuple from X^t exactly once.

A construction of TD$(3,5,v)$ via $(v,4,1)$-DM

Construction (Ji and Yin 2009)

If there is a $(v,4;1)$-DM, then there is a TD$(3,5,v)$.

- If $v \not\equiv 2 \pmod 4$ and $v \geq 4$, then there is a $(v,4;1)$-DM. (G. Ge, 2005)
- If $v \not\equiv 2 \pmod 4$ and $v \geq 4$, then there is a TD$(3,5,v)$.

Problem 1: Is there a TD$(3,5,v)$ for $v \equiv 2 \pmod 4$?

Difference Matrix Associated with an Adder

Let $D = (d_{ij})$ be a $(v, 4; 1)$-DM over an abelian group G. An v-tuple $s = (s_1, s_2, \ldots, s_v)$ over G is called an *adder* of the difference matrix D if $\{s_1, s_2, \ldots, s_v\} = G$ and the matrix

$$D^s = (d'_{ij}),\; d'_{ij} = d_{ij} \text{ for } i \in \{1, 2\}, \text{ and}$$
$$d'_{ij} = d_{ij} + s_j \text{ for } i \in \{3, 4\},$$

is also a $(v, 4; 1)$-DM over the group G.

A construction of TD$(3,6,v)$ via $(v,4;1)$-DM

Construction (Ji and Yin, JCTA 2010)

If there is a $(v,4;1)$-DM associated with an adder, then there is a TD$(3,6,v)$.

Theorem (Ji and Yin, JCTA 2010)

Let v be a positive integer which satisfies gcd$(v,4) \neq 2$ and gcd$(v,9) \neq 3$. Then there is a TD$(3,6,v)$.

Problem 2: Construct a $(3p,4;1)$-DM associated with an adder for any prime $p \geq 11$?

Suppose that there exist
(1) a TD$(3, k+2, g)$;
(2) a TD$(3, k, m)$;
(3) an ITD$((3,1), k, (m+m_i, m_i))$ for $i = 1, 2$;
(4) an ITD$((3,2), k, (m+m_1+m_2, m_1+m_2))$ for $i = 1, 2$;
(5) a TD$(3, k, m+m_1+m_2)$ or a TD$(3, k, m_1+m_2)$.
Then there exists a TD$(3, k, mg+m_1+m_2)$ that contains a TD$(3, k, m+m_1+m_2)$ and a TD$(3, k, m)$ as sub-designs.

New TD$(3, 5, 4n+2)$ (Yin et al, JCTA 2011)

Let x be an arbitrary odd positive integer. Let g be an arbitrary positive integer whose prime-power factors are all ≥ 7 such that $g \equiv 3 \pmod 4$. Then

(1) there is a TD$(3, 5, v)$ with $v = 35xg + 5 \equiv 2 \pmod 4$, if $x \equiv 1 \pmod 4$;
(2) there is a TD$(3, 5, v)$ with $v = 35xg + 7 \equiv 2 \pmod 4$, if $x \equiv 3 \pmod 4$.

Packing Number

The packing number $\mathrm{PN}(n, g, k, t)$ is the maximum number of blocks in any $\mathrm{HP}(n, g, k, t)$.

An $\mathrm{HP}(n, g, k, t)$ is called optimal if it has $\mathrm{PN}(n, g, k, t)$ blocks.

In 2001, Yin determined packing numbers $\mathrm{PN}(n, g, 3, 2)$:

$$PN(n, g, 3, 2) = \begin{cases} \lfloor \frac{ng}{3} \lfloor \frac{(n-1)g}{2} \rfloor \rfloor - 1, & \text{if } (n-1)g \equiv 0 \pmod 2 \\ & \text{and } n(n-1)g^2 \equiv -1 \pmod 3 \\ \lfloor \frac{ng}{3} \lfloor \frac{(n-1)g}{2} \rfloor \rfloor, & \text{otherwise.} \end{cases}$$

Optimal HP$(n, g, 4, 3)$ with even g

Theorem (Preprint)

Let n, g be positive integers with $n \equiv 0 \pmod 3$, $g \equiv 0 \pmod 2$ with $(n, g) \notin \{(a, b) : a \equiv 3 \pmod 6, b \in \{26, 38\}\}$. There is an optimal H$(n, g, 4, 3)$ packing design with $\frac{ng(n^2g^2-3ng+2g^2-8)}{24}$ blocks.

Proof: Start with an SQS, or a G-design, or an H$(n, 2, \{4, 6\}, 3)$. Input H-designs, Lattice H-designs and optimal HPs. An optimal H$(n, g, 4, 3)$ packing design is obtained.

Optimal HP$(n, 3, 4, 3)$ with $n \equiv 1 \pmod 8$

Theorem (Preprint)

For any $n \equiv 1 \pmod 8$ with $n \neq 17$, there is an optimal H$(n, 3, 4, 3)$ packing design.

Proof: Start with an SQS, or a G-design. Input a pair of matching CQSs and an optimal HP$(9, 3, 4, 3)$. An optimal H$(n, 3, 4, 3)$ packing design is obtained.

Future research

- Give more constructions for H$(m, g, k, 3)$.
- Give a direct construction of Steiner system with strength $t \geq 6$.
- Construct optimal HP$(n, g, 4, 3)$ for odd g.
- Give an infinite class of Steiner systems with $t \geq 4$.

Thank you!