Some Spectral extremal results for hypergraphs

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- Two vertices x and y are said to be adjacent, if there is an edge that contains both of these vertices.
- The degree of a vertex v, which is denoted by $d(v)$, is defined as the number of edges containing v.

- In 2005, Qi and Lim independently introduced the concept of eigenvalues for tensors.

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- An kth-order n-dimensional real tensor $\mathcal{T} = (\mathcal{T}_{i_1 \cdots i_k})$ consists of n^k real entries $\mathcal{T}_{i_1 \cdots i_k}$ for $1 \leq i_1, i_2, \cdots, i_k \leq n$.

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  a vector of dimension n is a tensor of order 1
  and a matrix is a tensor of order 2.
- $\mathcal{T}$ is called symmetric if the value of $\mathcal{T}_{i_1 \cdots i_k}$ is invariant under any permutation of its indices $i_1, i_2, \cdots, i_k$.

- Given a vector $x \in R^n$, $\mathcal{T}x^k$ is a real number and $\mathcal{T}x^{k-1}$ is an n-dimensional vector defined as follows.

$$\mathcal{T}x^k = \sum_{i_1,i_2,\cdots,i_k \in [n]} \mathcal{T}_{i_1 i_2 \cdots i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

and the ith component of $\mathcal{T}x^{k-1}$ is given by:

$$(\mathcal{T}x^{k-1})_i = \sum_{i_2,\cdots,i_k \in [n]} \mathcal{T}_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k}.$$

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- For some $\lambda \in \mathbb{C}$, if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying

$$\mathcal{T}x^{k-1} = \lambda x^{[k-1]}.$$

Then λ is an eigenvalue of $\mathcal{T}$ and x is its corresponding eigenvector.

- The maximal absolute value of the eigenvalues of $\mathcal{T}$ is called the spectral radius of $\mathcal{T}$, denoted by $\rho(\mathcal{T})$.

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  In 2012, Cooper and Dutle defined the adjacency tensor of a k-uniform hypergraph.

- The adjacency tensor $\mathcal{A}(H)$ is a k-th order n-dimensional symmetric tensor, where:

$$\mathcal{A}(H)_{i_1\cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{i_1, \cdots, i_k\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Contents

Problems

For a fixed family $\mathcal{F}$, recall that the classical Turán problem is of the following type:

Problem A. What is the maximum number of edges of a graph of order n*, not containing a given* $\mathcal{F}$*?*

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For a fixed family $\mathcal{F}$, recall that the classical Turán problem is of the following type:

Problem A. What is the maximum number of edges of a graph of order n, not containing a given $\mathcal{F}$?

The following is the natural spectral analog to the Turán problem of graphs:

Problem B. What is the maximum spectral radius of a graph of order n, not containing a given $\mathcal{F}$?

Spectral versions on Turán problems

Let $\lambda(G)$ be the spectral radius of the adjacency matrix of the graph G.

Theorem 1 (Guiduli, 1998; Nikiforov, 2002)

If G is a graph of order n with no complete subgraph of order $r+1$, then $\lambda(G) \leq \lambda(T_r(n))$. Equality holds if and only if $G = T_r(n)$.

Corollary 2 (Spectral version of Mantel's Theorem)

If G is a graph of order n without C_3, then $\lambda(G) \leq \lceil \frac{n}{2} \rceil$. Equality holds if and only if $G = T_2(n)$.

Spectral versions on Turán problems

Theorem 3 (Nikiforov, 2007)

If G is a graph of order n without C_4, then

$$\lambda^2(G) - \lambda(G) \leq n - 1.$$

Equality holds if and only if every two vertices of G have exactly one common neighbor, i.e., when G is the friendship graph.

Theorem 4 (Favaron, Mahéo and Saclé, 1993)

If G is a graph of order n with neither C_3 nor C_4, then $\lambda(G) \leq \sqrt{n-1}$.

Turán extremal problem on hypergraphs

- Given a fixed k-uniform family $\mathcal{F}$, the Turán number of $\mathcal{F}$, denoted by $ex_k(n, \mathcal{F})$, is the maximum number of edges of an $\mathcal{F}$-free hypergraph on n vertices.

Turán extremal problem on hypergraphs

- Given a fixed k-uniform family $\mathcal{F}$, the Turán number of $\mathcal{F}$, denoted by $ex_k(n, \mathcal{F})$, is the maximum number of edges of an $\mathcal{F}$-free hypergraph on n vertices.
- A hypergraph H is called linear if every two edges have at most one vertex in common. Given a family of k-uniform linear hypergraphs $\mathcal{F}$, the linear Turán number of $\mathcal{F}$, denoted by $ex_k^{lin}(n, \mathcal{F})$, is the maximum number of edges in an $\mathcal{F}$-free k-uniform linear hypergraph on n vertices.

What is the maximum spectral radius of the adjacency tensor of a uniform hypergraph of order n, not containing a given $\mathcal{F}$?

Contents

Useful tools

- For a vertex v, let N_v be the neighborhood of v, i.e., $N_v = \{x \in V \setminus \{v\} | v, x \in e \text{ for some } e \in E\}$.

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- The codegree of two vertices u and v, denoted by $d(u, v)$, is the number of edges containing both u and v.
- For a set $X \subseteq V$, let $E_t(X) = \{e | e \in E \text{ and } |e \cap X| = t\}$ and $e_t(X)$ be the number of edges in $E_t(X)$, respectively.

Useful tools

Lemma 5 (Hou, Chang and Cooper, 2019+)

Let H be a connected simple k-uniform hypergraph and ρ be the spectral radius of the adjacency tensor of H. Then

$$\rho^2 \leq \frac{1}{k-1} \sum_{t=1}^{k} \sum_{e \in E_t(N_u)} \sum_{v \in N_u \cap e} d(u,v) \qquad (1)$$

where u is the vertex corresponding to a maximum entry of the principal eigenvector.

Note that above Lemma illustrates a relationship between spectral radius of the adjacency tensor and structural properties of hypergraphs.

Useful tools

It is clear that the codegree of each pair of adjacent vertices in H is exactly 1 if H is a linear hypergraph.

Corollary 6 (Hou, Chang and Cooper, 2019+)

Let H be a connected simple k-uniform linear hypergraph and ρ be the spectral radius of the adjacency tensor of H. Let u be the vertex with maximum eigenvector entry. Then
(1) $\rho^2 \leq \frac{1}{k-1}[e_1(N_u) + 2e_2(N_u) + \cdots + ke_k(N_u)]$;

(2) $\rho^2 \leq \frac{1}{k-1} \sum_{v \in N_u} d(v)$.

Linear hypergraphs without Fan^k

Definition 7 (Mubayi and Pikhurko, 2007)

For $k \geq 2$, the k-fan Fan^k is the k-uniform linear hypergraph having k edges $f_1, f_2, \cdots, f_k$ pairwise intersecting in the same vertex v and an additional edge g intersecting all f_i in a vertex different from v.

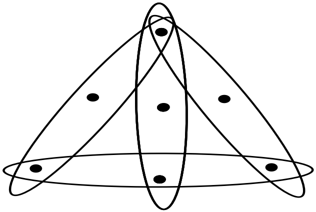

Figure: Fan^3

Linear hypergraphs without Fan^k

Theorem 8 (Füredi, Gyárfás, 2017)

One has $ex_k^{lin}(n, Fan^k) \leq \frac{n^2}{k^2}$ for all $k \geq 2$. The only extremal hypergraphs are the transversal designs on n vertices with k groups.

Theorem 9 (Hou, Chang and Cooper, 2019+)

Let $\mathcal{H}$ denote the set of linear k-uniform hypergraphs of order $n(n \equiv 0 \ mod(k))$ with forbidden Fan^k and ρ be the maximum spectral radius of hypergraphs in $\mathcal{H}$. For n sufficiently large, we have $\rho = \frac{n}{k}$.

- After using the well-known inequality $\rho \geq \frac{km}{n}$, this result implies $m \leq \frac{n^2}{k^2}$.

Definition 10 (Gerbner and Palmer, 2017)

Let $F = (V(F), E(F))$ be a graph and $\mathcal{B} = (V(\mathcal{B}), E(\mathcal{B}))$ be a hypergraph. We say $\mathcal{B}$ is Berge F if there is a bijection $\phi : E(F) \to E(\mathcal{B})$ such that $e \subseteq \phi(e)$ for all $e \in E(F)$. In other words, given a graph F, we can obtain a Berge F by replacing each edge of F with a hyperedge that contains it.

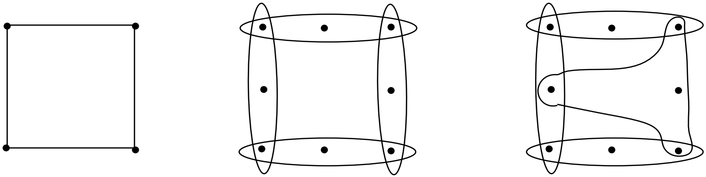

Figure: C_4 and Berge C_4

Linear hypergraphs without Berge C_4

Theorem 11 (Ergemlidze, Győri and Methuku, 2018)

$ex_3^{lin}(n, \{C_4\}) \le \frac{1}{6} n^{\frac{3}{2}} + O(n)$.

Theorem 12 (Hou, Chang and Cooper, 2019+)

Let $\mathcal{H}$ denote the set of linear k-uniform hypergraphs of order n with forbidden Berge C_4 and ρ be the maximum spectral radius among hypergraphs in $\mathcal{H}$. For n sufficiently large, we have $\rho \le \frac{\sqrt{n}}{k-1} + O(1)$.

- After using the well-known inequality $\rho \ge \frac{km}{n}$, this result implies $m \le \frac{1}{k(k-1)} n^{\frac{3}{2}} + O(n)$.

Linear hypergraphs with girth at least five

Theorem 13 (Lazebnik and Verstraëte, 2003)

$ex_3^{lin}(n, \{C_3, C_4\}) = \frac{1}{6}n^{\frac{3}{2}} + O(n)$.

Theorem 14 (Hou, Chang and Cooper, 2019+)

Let $\mathcal{H}$ denote the set of linear k-uniform hypergraphs of order n with neither Berge C_3 nor Berge C_4 and ρ be the maximum spectral radius of hypergraphs in $\mathcal{H}$. For n sufficiently large, we have $\rho \leq \frac{\sqrt{n}}{k-1} + O(1)$.

- After using the well-known inequality $\rho \geq \frac{km}{n}$, this result implies $m \leq \frac{1}{k(k-1)}n^{\frac{3}{2}} + O(n)$.

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Thank You For Your Attention!