The Product Version of Erdős-Ko-Rado Theorem

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Introduction

Theorem (EKR Theorem)

If $\mathcal{A}$ is an intersecting family of k-subsets of $[n] = \{1, 2, \ldots, n\}$, i.e., $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A}$, then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

subject to $n \geq 2k$. Equality holds if and only if every subset in $\mathcal{A}$ contains a common element of $[n]$ except for $n = 2k$.

- P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser., 2 (1961), 313-318.

Introduction

Suppose that $n = n_1 + \ldots + n_d$, $k = k_1 + \ldots + k_d$, and $X = X_1 \cup \ldots \cup X_d$ with $|X_i| = n_i$. Define

$$\mathcal{F} = \left\{ A \in \binom{X}{k} : |A \cap X_i| = k_i \text{ for } 1 \leq i \leq d \right\} = \binom{X_1}{k_1} \times \cdots \times \binom{X_d}{k_d}.$$

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Theorem (Direct-product Theorem on EKR Theorem)

Suppose that $\mathcal{A}$ is an intersecting family of $\mathcal{F}$ and $\frac{1}{2} \geq \frac{k_1}{n_1} \geq \ldots \geq \frac{k_d}{n_d}$. Then

$$\frac{|\mathcal{A}|}{|\mathcal{F}|} \leq \frac{k_1}{n_1}.$$

- P. Frankl, Erdős-Ko-RAdo Theorem for Direct Products, EuJC 17 (1996) 727-730.

Introduction

Theorem

Let $2 \leq n_1 = \cdots = n_p < n_{p+1} \leq \ldots \leq n_q$, $1 \leq p \leq q$. *Let* $G = S_{n_1} \times \ldots \times S_{n_q}$ *be the direct products of the symmetric group* S_{n_i} *on* $[n_i]$. *Suppose* $\mathcal{A}$ *is an intersecting family in* G. *Then* $|\mathcal{A}| \leq (n_1 - 1)! \prod_{2 \leq i \leq q} n_i!$. *Moreover, except for the following cases:*

(i) $n_1 = \cdots = n_p < n_{p+1} = 3 \leq n_{p+2} \leq \cdots \leq n_q$;

(ii) $n_1 = n_2 = 3 \leq n_3 \leq \cdots \leq n_q$;

(iii) $n_1 = n_2 = n_3 \leq n_4 \leq \cdots \leq n_q$,

equality holds if and only if $\mathcal{A} = \{(\alpha_1, \ldots, \alpha_p) : \alpha_i(x) = y\}$ *for some* $i \in [p]$ *and* $x, y \in [n_i]$.

- C.Y. Ku and T.W.H. Wong, Intersecting families in the alternating group and direct

Introduction

- Kneser graph $K_{n,k}$: vertex set $\binom{[n]}{k}$, $A \sim B$ iff $A \cap B = \emptyset$.
- $\alpha(K_{n,k}) = \binom{n-1}{k-1}$.

Introduction

Definition

The direct product of $G \times H$ of two graphs G and H is defined by

$$V(G \times H) = \{(u, v) : u \in V(G) \text{ and } v \in V(H)\}$$

and

$$(u_1, v_1) \sim (u_2, v_2) \text{ iff } u_1 \sim u_2 \text{ and } v_1 \sim v_2.$$

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$$\alpha(G \times H) \geq \max\{\alpha(G)|H|, \alpha(H)|G|\}$$

Introduction

Theorem (H.J. Zhang 2012)

If G and H are vertex-transitive, then

$$\alpha(G \times H) = \max\{\alpha(G)|H|, \alpha(H)|G|\}.$$

Introduction

Let G be a graph, the set of all all independent set of G denoted by $I(G)$. For $u \in V(G)$, set $I_u(G) = \{S \in I(G) : u \in I\}$.

- *fractional coloring* f: a map from $I(G)$ to $[0, 1]$ with $\sum_{S \in I_u(G)} f(S) = 1$.
- *total weight*: $\omega(f) = \sum_{s \in I(G)} f(S)$.
- *fractional chromatic number* $\chi_f(G)$: the minimum total weight of a fractional coloring of G.

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Conjecture

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$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

Introduction

Definition

The *tensor product* (G_1, G_2, G_3) of G_1, G_2 and G_3 is defined by

$$V(G_1, G_2, G_3) = V(G_1) \times V(G_2) \times V(G_3) = \{(u_1, u_2, u_3) : u_i \in V(G_i)\}$$

and

$$E(G_1, G_2, G_3) = \left\{ \left((u_1, u_2, u_3), (v_1, v_2, v_3)\right) : |\{i : (u_i, v_i) \in E(G_i)\}| \geq 2 \right\}.$$

Introduction

By definition, $I_1 \times I_2 \times V(G_3)$ is an independent set of (G_1, G_2, G_3) if $I_i \in I(G_i)$, $i = 1, 2$. Hence

$$\alpha(G_1, G_2, G_3) \geq \max\{\alpha(G_1)\alpha(G_2)|G_3|, \alpha(G_1)\alpha(G_3)|G_2|, \alpha(G_2)\alpha(G_3)|G_1|\}.$$

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Problem

For three vertex-transitive graphs G_1, G_2 and G_3, does

$$\alpha(G_1, G_2, G_3) = \max\{\alpha(G_1)\alpha(G_2)|G_3|, \alpha(G_1)\alpha(G_3)|G_2|, \alpha(G_2)\alpha(G_3)|G_1|\}$$

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always hold?

Shift operation

Definition

- Let $\mathcal{A}$ be a family in $\binom{[n]}{k}$. For $i, j \in [n]$ with $i < j$ and $A \in \mathcal{A}$, define

$$s_{ij}(A) = \begin{cases} (A \setminus \{j\}) \cup \{i\}, & \text{if } j \in A,\ i \notin A, (A \setminus \{j\}) \cup \{i\} \notin \mathcal{A}; \\ A, & \text{otherwise,} \end{cases}$$

and let $s_{ij}(\mathcal{A}) = \{s_{ij}(A) : A \in \mathcal{A}\}$.

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- $|s_{ij}(\mathcal{A})| = |\mathcal{A}|$ and $s_{ij}(\mathcal{A})$ is also an intersecting family if $\mathcal{A}$ is so.
- $\mathcal{A}$ is called *left-compressed* if $s_{ij}(\mathcal{A}) = \mathcal{A}$ for all $i < j$.

Proof of EKR theorem

Proof. Let $\mathcal{A}$ be a maximum intersecting family of $\binom{[n]}{2k}$ $(n > 2k)$ with $s_{ij}(\mathcal{A}) = \mathcal{A}$ for all $1 \leq i < j \leq n$.

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Set $\mathcal{A}_0 = \{A \in \mathcal{A} : n \notin A\}$, $\mathcal{A}_1 = \{A \in \mathcal{A} : n \in A\}$ and $\mathcal{A}_1' = \{A - n : A \in \mathcal{A}_1\}$.

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$\mathcal{A}_1'$ is also an intersecting family of $\binom{[n-1]}{k-1}$, $|\mathcal{A}_1'| \le \binom{n-2}{k-2}$ by induction.

Therefore,

$$|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1| = |\mathcal{A}_0| + |\mathcal{A}_1'| \le \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}.$$

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Arrange the elements of $[n]$ on a cycle,

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Circular clique

- *Circular clique* $K_{n:k}$: vertex set $\mathcal{X}_{n,k}$, $A \sim B$ iff $A \cap B = \emptyset$.
- $\alpha(K_{n,k}) = k$.

Operation

Let S be a subset of $G \times K_{n:k}$. For $u \in V(G)$, set

$$\partial_u(S) = \{i \in [n] : (u, i) \in S\}.$$

Let I be a fixed maximum independent set of $K_{n:r}$. The subset $\triangle_I(S)$ of $V(G) \times V(K_{n:r})$ is defined by:

$$\partial_u(\triangle_I(S)) = \begin{cases} I, & \text{if } 0 < |\partial_u(S)| < n; \\ \partial_u(S), & \text{otherwise,} \end{cases}$$

for $u \in V(G)$.

Appication

Lemma (Geng,Wang and Zhang)

Let G be a vertex-transitive graph and I a maximum independent set of $K_{n:k}$. If S is a maximum independent set of $G \times K_{n:k}$, then $\triangle_I(S)$ is also a maximum independent set of $G \times K_{n:k}$.

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$$\frac{|B|}{|N[B]|} = \frac{k}{n} = \frac{\alpha(G)}{|V(G)|}$$

Application

Lemma (Albertson and Collins)

Let G and H be two graphs such that G is vertex-transitive and there exists a homomorphism $\phi : H \mapsto G$. Then $\frac{\alpha(G)}{|V(G)|} \leq \frac{\alpha(H)}{|V(H)|}$, and equality holds if and only if for any independent set I of cardinality $\alpha(G)$ in G, $\phi^{-1}(I)$ is an independent set of cardinality $\alpha(H)$ in H.

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Lemma

$\frac{\alpha(G)}{|V(G)|} \leq \frac{\alpha(G[B])}{|B|}$ *holds for all* $B \subseteq V(G)$. *Equality implies that* $|I \cap B| = \alpha(G[B])$ *for every maximum independent set* I *of* G.

Application

Let $X_1, \ldots, X_m$ be m pairwise disjoint sets with the same size n, and let $k_1, k_2, \ldots, k_m$ be positive integers with $k_i \leq n/2$. $X = X_1 \cup \cdots \cup X_m$ and $k = k_1 + \cdots + k_m$. Set

$$\begin{aligned}
\Omega_m(n; k_1, \ldots, k_m) &= \left\{ A \in \binom{X}{k} : \{|A \cap X_1|, \ldots, |A \cap X_m|\} = \{k_1, \ldots, k_m\} \right\} \\
&= \bigcup_{\sigma \in S_m} \left\{ A_1 \cup \cdots \cup A_m : A_i \in \binom{X_i}{k_{\sigma(i)}}, i = 1, \ldots, m \right\} \\
&= \bigcup_{\sigma \in S_m} \binom{X_1}{k_{\sigma(1)}} \times \cdots \times \binom{X_m}{k_{\sigma(m)}}.
\end{aligned}$$

Application

$$\Omega' = \bigcup_{\sigma \in S_m} K_{n:k_{\sigma(1)}} \times \cdots \times K_{n:k_{\sigma(m)}}.$$

Let I_j be a fixed maximum independent set of $K_{n:k_j}$, and $\cup I_j$ is an independent set of $\cup K_{n:k_j}$.

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$$\Omega' = \bigcup_{\sigma \in S_m} K_{n:k_{\sigma(1)}} \times \cdots \times K_{n:k_{\sigma(m)}}.$$

Let I_j be a fixed maximum independent set of $K_{n:k_j}$, and $\cup I_j$ is an independent set of $\cup K_{n:k_j}$.Let S be a maximum independent set of Ω'.

$S_\sigma = S \cap K_{n:k_{\sigma(1)}} \times \cdots \times K_{n:k_{\sigma(m)}}$.

$$\triangle_i(S) = \bigcup_{\sigma \in S_m} \triangle_{I_{\sigma(i)}}(S_\sigma)$$

Application

Lemma

Let S be a maximum independent set of Ω', then $\triangle_i(S)$ is also a maximum independent set of Ω' for all $1 \leq i \leq m$.

Application

$\triangle_i(S) = S$

$$T_1 \times T_2 \times \cdots \times T_m, T_i = I \text{ or } K_{n:k_j} \backslash I$$

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Theorem (Wang and Zhang SIAM 2018)

Let n and $k_1, \ldots, k_m$ be positive integers satisfying $n \geq 2k_i$. If $\mathcal{F}$ is an intersecting family in $\Omega_m(n; k_1, \ldots, k_m)$, then

$$|\mathcal{F}| \leq \frac{k}{nm} p(k_1, \ldots, k_m) \prod_{i=1}^{m} \binom{n}{k_i},$$

where $k = k_1 + \cdots + k_m$. Also equality holds if and only if $\mathcal{F} = \{A \in \Omega : a \in A\}$ for some $a \in X$ except for $k_1 = \cdots = k_m = \frac{n}{2}$.

Application

Theorem

Let G and H be two vertex-transitive graphs and $K_{n:r}$ be a circular graph. Then

$$\alpha(G, H, K_{n:r}) = \max\{\alpha(G)\alpha(H)n, \alpha(G)r|H|, r\alpha(H)|G|\}.$$

Application

Theorem (Xiao, Zhang and Zhang Disc 2018)

Let G and H be two graphs and $K_{n:r}$ be a circular graph. Then

$$\chi_f(G, H, K_{n:r}) = \min\{\chi_f(G)\chi_f(H), \chi_f(G)r, r\chi_f(H)\}.$$

Many Thanks!