Cliques in graphs with given matching number

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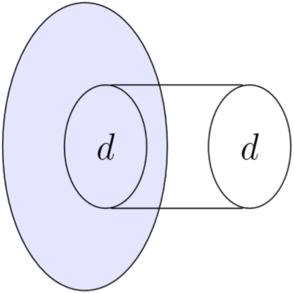
- A hard one: $\mathrm{ex}(n, C_3, C_5)$.

Example 1: nonhamiltonian graphs

- Erdős showed if G is nonhamiltonian and $\delta(G) \geq k$, then $e(G) \leq \max\{h(n,k), h(n, \lfloor \frac{n-1}{2} \rfloor\}$, here $h(n,k) = \binom{n-k}{2} + k^2$.



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- For n large enough and a general graph F, $N(G,F) \leq N(H_{n,k}, F)$.

Example 2: no long cycles or paths

Erdős and Gallai

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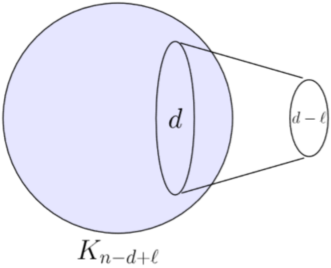
- Füredi, Kostochka, and Luo: Assume $0 \le l < k \le \lfloor \frac{n+l-1}{2} \rfloor$. If G is an n vertex graph with minimum degree $\delta(G) \ge k$, and G is not ℓ-hamiltonian, then $N_s(G) \le \max\{h_s(n,k,\ell), h_s(n, \lfloor \frac{n+l-1}{2} \rfloor, \ell)\}$



t–stable properties

Let $\mathcal{P}$ be a graph property. If whenever $G + uv$ has the property $\mathcal{P}$ and $d_G(u) + d_G(v) \geq t$, then G itself also has the property $\mathcal{P}$.

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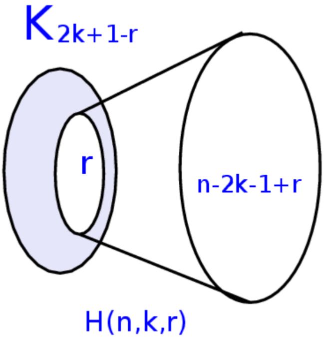
Question (Füredi, Kostochka, and Luo)

Determine the maximum number of copies of cliques in graphs having a stable property $\mathcal{P}$.

Our results



Let $h_s(n,k,r) = \binom{2k+1-r}{s} + (n-2k-1+r)\binom{r}{s-1}$.

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Erdős and Gallai:

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Theorem (Duan, Ning, P., Wang, and Yang)

If G is a graph with $n \geq 2k+2$ vertices, minimum degree δ, and $\nu(G) \leq k$, then $N_s(G) \leq \max\{h_s(n,k,\delta), h_s(n,k,k)\}$ for each $s \geq 2$.

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- G' is the $(2k+1)$-closure and $d_{H_1}(u), d_{H_1}(v) \geq k+1$.

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- Step 2: consider the $(2k - t + 2)$-core H_2 of G'.
- Claim $H_1 \subsetneq H_2$.
- If $H_1 = H_2$, then $N_s(G') \leq \binom{t}{s} + (n - t)\binom{2k-t+1}{s-1} = h_s(n, k, 2k + 1 - t)$.

Proof sketch: III

- By the convexity of $h_s(n, k, r)$ and $\delta \leq 2k - t + 1 \leq k$, we get $N_s(G') \leq \max\{h_s(n, k, \delta), h_s(n, k, k)\}$.

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- Note $d_{H_1}(u) \ge t - 1$ and $d_{H_2}(v) \ge 2k - t + 2$.
- Then $d'_G(u) + d'_G(v) \ge 2k + 1$ and they are adjacent as G' is the $(2k+1)$-closure.

Known stability results: I

Theorem (Erdős)

Let n and d be integers with $1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$. If G is nonhamiltonian and $\delta(G) \geq k$, then $e(G) \leq \max\{h(n,k), h(n, \lfloor \frac{n-1}{2} \rfloor)\}$.

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Question: Can we establish a stability version of the result on the number of copies of general graphs?

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Theorem (Erdős and Gallai)

If G does not contain a cycle of length at least k, then $e(G) \leq \frac{n-1}{k-2}\binom{k-1}{2}$.

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Theorem (Kopylov)

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Theorem (Füredi, Kostochka, and Verstraëte)

Assume $n \geq \frac{3k}{2}$. If G is 2-connected, does not contains a cycle of length at least k, and $e(G) > h(n,k,\lfloor\frac{k-1}{2}\rfloor) - 1)$, then G is in a family of graphs.

Known stability results: IV

Theorem (Füredi, Kostochka, Luo, and Verstraëte)

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Theorem (Ma and Ning)

If G is 2-connected, $\delta(G) \geq d$, does not contains a cycle of length at least k, and
$e(G) > \max\{h(n,k,d+1), h(n,k,\lfloor\frac{k}{2}\rfloor - 1)\}$, then G is in a family of graphs.

Our result: stability versions

Theorem (Duan, Ning, P., Wang, and Yang)

Let G be a graph on n vertices with $\delta(G) \geq \delta$ and $\nu(G) \leq k$. If $n \geq 2k+2$ and $e(G) > \max\{h_2(n,k,\delta), h_2(n,k,k-2)\}$, then G has to be a subgraph of $H(n,k,k)$ or $H(n,k,k-1)$.

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Remark: Observe $h_s(n,k,k) - h_s(n,k,k-2) = \Theta(h_s(n,k,k))$.

Our result: stability versions

Theorem (Duan, Ning, P., Wang, and Yang)

Let G be a graph on n vertices with $\delta(G) \geq \delta$ and $\nu(G) \leq k$. If $n \geq 2k+2$ and $e(G) > \max\{h_2(n,k,\delta), h_2(n,k,k-2)\}$, then G has to be a subgraph of $H(n,k,k)$ or $H(n,k,k-1)$.

Remark: Observe $h_s(n,k,k) - h_s(n,k,k-2) = \Theta(h_s(n,k,k))$.

Theorem (Duan, Ning, P., Wang, and Yang)

Let G be a graph on n vertices with $\delta(G) \geq \delta$ and $\nu(G) \leq k$. If $n \geq 2k+11$ and $e(G) > \max\{h_2(n,k,\delta+2), h_2(n,k,k)\}$, then G has to be a subgraph of $H(n,k,\delta)$ or $H(n,k,\delta+1)$.

Thanks