Signed Mahonian Identities on Permutations with Subsequence Restrictions

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Outline

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Preliminaries

- $\mathfrak{S}_n = \{\text{permutations of } \{1, 2, \ldots, n\}\}$.
- For $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$:
  - $\mathsf{inv}(\sigma) = |\{(i, j) : i < j \text{ and } \sigma_i > \sigma_j\}|$,
  - $\mathsf{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$,
  - $\mathsf{des}(\sigma) = |\mathsf{Des}(\sigma)|$,
  - $\mathsf{maj}(\sigma) = \sum_{i\in \mathsf{Des}(\sigma)} i$.

Example

$\sigma = 31452 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}$,

$\mathsf{inv}(\sigma) = 4$, $\mathsf{Des}(\sigma) = \{1, 4\}$, $\mathsf{des}(\sigma) = 2$, $\mathsf{maj}(\sigma) = 5$.

Signed Major Distribution

- MacMahon proved

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{inv}(\sigma)} = [2]_q[3]_q\cdots[n]_q,$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$.

- Gessel and Simion obtained

$$\sum_{\sigma\in\mathfrak{S}_n} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = [2]_{-q}[3]_q\cdots[n]_{(-1)^{n-1}q}.$$

Signed Major Distribution

Example

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$\sum_{\sigma\in\mathfrak{S}_3} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_3} q^{\mathsf{inv}(\sigma)} = [2]_q[3]_q = (1+q)(1+q+q^2) = 1+2q+2q^2+q^3.$$

$$\sum_{\sigma\in\mathfrak{S}_3} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = [2]_{-q}[3]_q = (1-q)(1+q+q^2) = 1-q^3.$$

Permutations with Restrict Subsequence

- We consider only the words without repeated letters on a set.
- Given a word $W = w_1 w_2 \cdots w_k$ on the set $\{1, 2, \ldots, n\}$, let

$$\begin{aligned}\mathfrak{S}_n(W) &= \{\sigma \in \mathfrak{S}_n : W \text{ is a subsequence of } \sigma\} \\ &= \{\sigma \in \mathfrak{S}_n : \sigma^{-1}(w_1) < \sigma^{-1}(w_2) < \cdots < \sigma^{-1}(w_k)\}.\end{aligned}$$

- For $a, b \in \{1, 2, \cdots, n\}$ and $a < b$, let

$$\mathfrak{S}_n(a:b) = \mathfrak{S}_n(a(a+1)\cdots b).$$

Example

$\mathfrak{S}_4(34) =$
$\{1234, 1324, 1342, 2134, 2314, 2341, 3124, 3142, 3214, 3241, 3412, 3421\}.$

Signed Major Distribution with Subsequence Restriction

- Stanley, Foata and Schützenberger proved

$$\sum_{\sigma\in\mathfrak{S}_n(n-k+1:n)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n(n-k+1:n)} q^{\mathsf{inv}(\sigma)} = [k+1]_q[k+2]_q\cdots[n]_q.$$

- Caselli obtained

$$\sum_{\sigma\in\mathfrak{S}_n(n-k+1:n)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = [k+1]_{(-1)^{nk+n+k}q}[k+2]_{(-1)^{k+1}q}\cdots[n]_{(-1)^{n-1}q}.$$

Example

$\mathfrak{S}_4(34)$	inv	maj	$\mathfrak{S}_4(34)$	inv	maj
1234	0	0	3124	2	1
1324	1	2	3142	3	4
1342	2	3	3214	3	3
2134	1	1	3241	4	4
2314	2	2	3412	4	2
2341	3	3	3421	5	5

$$\sum_{\sigma\in\mathfrak{S}_4(3:4)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_4(3:4)} q^{\mathsf{inv}(\sigma)} = [3]_q[4]_q$$
$$= (1+q+q^2)(1+q+q^2+q^3) = 1+2q+3q^2+3q^3+2q^4+q^5.$$

$$\sum_{\sigma\in\mathfrak{S}_4(3:4)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = [3]_q[4]_{-q}$$
$$= (1+q+q^2)(1-q+q^2-q^3) = 1+q^2-q^3-q^5.$$

Caselli raised a question about giving a bijective proof of the following observation:

Conjecture (Caselli[2], 2012)

If n is even or k is odd then

$$\sum_{\sigma\in\mathfrak{S}_n(1:k)}(-1)^{\mathsf{inv}(\sigma)}q^{\mathsf{maj}(\sigma)}=\sum_{\sigma\in\mathfrak{S}_n(n-k+1:n)}(-1)^{\mathsf{inv}(\sigma)}q^{\mathsf{maj}(\sigma)}.$$

Our main goal is to solve this conjecture (and more).

Example

$\mathfrak{S}_4(12)$	inv	maj	$\mathfrak{S}_4(34)$	inv	maj
1234	0	0	3124	2	1
1243	1	3	3142	3	4
1324	1	2	3412	4	2
1342	2	3	4123	3	1
1423	2	2	4132	4	4
1432	3	5	4312	5	3

$\mathfrak{S}_4(34)$	inv	maj	$\mathfrak{S}_4(34)$	inv	maj
1234	0	0	3124	2	1
1324	1	2	3142	3	4
1342	2	3	3214	3	3
2134	1	1	3241	4	4
2314	2	2	3412	4	2
2341	3	3	3421	5	5

$$\sum_{\sigma \in \mathfrak{S}_4(1:2)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = 1 + q^2 - q^3 - q^5.$$

$$\sum_{\sigma \in \mathfrak{S}_4(3:4)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = 1 + q^2 - q^3 - q^5.$$

Outline

To solve Caselli's conjecture, we found a crucial relation between $\mathfrak{S}_n(W)$ and $\mathfrak{S}_n(W+2)$.

Theorem (Eu, Fu, Hsu, Liao, Sun[3], 2019)

For any word W on the set $\{1,2,\ldots,n-2\}$, there is a bijection $\phi:\mathfrak{S}_n(W)\to\mathfrak{S}_n(W+2)$ such that

$$\mathrm{Des}(\phi(\sigma))=\mathrm{Des}(\sigma) \ \textit{ and } \ \mathrm{inv}(\phi(\sigma))\equiv\mathrm{inv}(\sigma) \pmod 2.$$

Hence

$$\sum_{\sigma\in\mathfrak{S}_n(W)}(-1)^{\mathrm{inv}(\sigma)}t^{\mathrm{des}(\sigma)}q^{\mathrm{maj}(\sigma)}=\sum_{\sigma\in\mathfrak{S}_n(W+2)}(-1)^{\mathrm{inv}(\sigma)}t^{\mathrm{des}(\sigma)}q^{\mathrm{maj}(\sigma)}.$$

Example

$$\sum_{\sigma\in\mathfrak{S}_5(213)}(-1)^{\mathrm{inv}(\sigma)}t^{\mathrm{des}(\sigma)}q^{\mathrm{maj}(\sigma)}=\sum_{\sigma\in\mathfrak{S}_5(425)}(-1)^{\mathrm{inv}(\sigma)}t^{\mathrm{des}(\sigma)}q^{\mathrm{maj}(\sigma)}$$

For the relation between $\mathfrak{S}_n(W)$ and $\mathfrak{S}_n(W+1)$, we found the following result.

Theorem (Eu, Fu, Hsu, Liao, Sun[3], 2019)

For $2 \leq k \leq n-1$ *and* $1 \leq b \leq n-k$, *let*

$$U = b(b+1)\cdots(b+k-1) \text{ and } V = (b+1)(b+2)\cdots(b+k),$$

then there is a bijection $\gamma : (\mathfrak{S}_n(U) \setminus \mathfrak{S}_n(V)) \to (\mathfrak{S}_n(V) \setminus \mathfrak{S}_n(U))$ *such that*

$$\mathrm{Des}(\gamma(\sigma)) = \mathrm{Des}(\sigma) \text{ and } \mathrm{inv}(\gamma(\sigma)) - \mathrm{inv}(\sigma) \equiv k-1 \pmod 2.$$

The above results imply the following theorem.

Theorem (Eu, Fu, Hsu, Liao, Sun[3], 2019)

For $2 \le k \le n-1$ *and* $1 \le b \le n-k+1$, *we have*

k	n	b	$\sum_{\sigma\in\mathfrak{S}_n(b:b+k-1)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)}$
odd	*any*	*any*	$[k+1]_{(-1)^{nk+n+k}q}[k+2]_{(-1)^{k+1}q}\cdots[n]_{(-1)^{n-1}q}$
even *even*	*even* *odd*	*odd* *even*	$[k+1]_{(-1)^{nk+n+k}q}[k+2]_{(-1)^{k+1}q}\cdots[n]_{(-1)^{n-1}q}$
even *even*	*even* *odd*	*even* *odd*	$(2-[k+1]_{(-1)^{nk+n+k}q})[k+2]_{(-1)^{k+1}q}\cdots[n]_{(-1)^{n-1}q}$

This result gives a complete picture of Caselli's conjecture for all parity cases of n and k.

Conjecture (Caselli[2], 2012)

If n *is even or* k *is odd then*

$$\sum_{\sigma\in\mathfrak{S}_n(1:k)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n(n-k+1:n)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)}.$$

Han[5], and Haglung, Loehr, Remmel[4] derived an insertion lemma which describes the increment of major index resulting from the insertion of r in W: For any $r \in \{1, 2, \ldots, n\}$ and any permutation W on $\{1, 2, \ldots, n\} \setminus \{r\}$, we have

$$\sum_{\sigma \in \mathfrak{S}_n(W)} q^{\mathsf{maj}(\sigma)} = q^{\mathsf{maj}(\sigma)} [n]_q.$$

We derive an extension of the insertion lemma to a signed version.

Theorem (Eu, Fu, Hsu, Liao, Sun[3], 2019)

For any $r \in \{1, 2, \ldots, n-1\}$ and any permutation W of $\{1, 2, \ldots, n\} \setminus \{r, r+1\}$, we have

$$\sum_{\sigma \in \mathfrak{S}_n(W)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = (-1)^{\mathsf{inv}(W)} q^{\mathsf{maj}(W)} [n-1]_{(-1)^{n-2}q} [n]_{(-1)^{n-1}q}.$$

Outline

Lemma (Eu, Fu, Hsu, Liao, Sun[3], 2019)

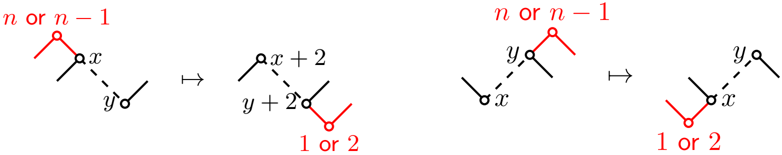
For any permutation W on the set $\{1, 2, \ldots, n-2\}$, there is a bijection $\phi : \mathfrak{S}_n(W) \to \mathfrak{S}_n(W+2)$ such that

$$\mathrm{Des}(\phi(\sigma)) = \mathrm{Des}(\sigma) \;\; \textit{and} \;\; \mathrm{inv}(\phi(\sigma)) \equiv \mathrm{inv}(\sigma) \pmod 2.$$

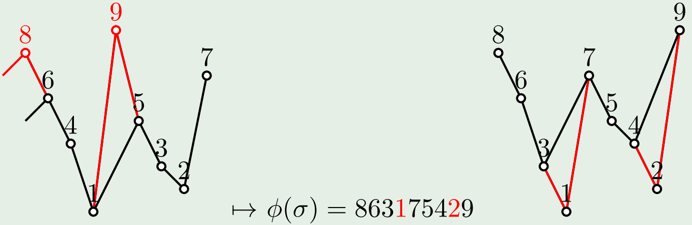
To keep descent set and the parity of inversion under this bijection, we take the following steps:

- Remove $n-1$ and n.
- Add the other numbers by 2.
- Use descent set to find the appropriate positions to insert 1 and 2.
- Use the parity of inversion to decide the ordering of 1 and 2.

- Basic strategy:



Example



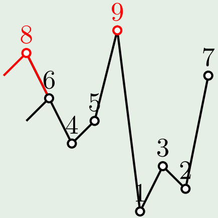
$\sigma = 864195327 \mapsto \phi(\sigma) = 863175429$

$\mathsf{Des}(\phi(\sigma)) = \mathsf{Des}(\sigma) = \{1, 2, 3, 5, 6, 7\}$

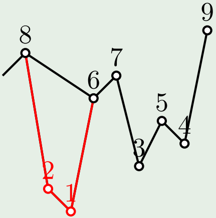
$\mathsf{inv}(\phi(\sigma)) \equiv \mathsf{inv}(\sigma) \pmod 2$

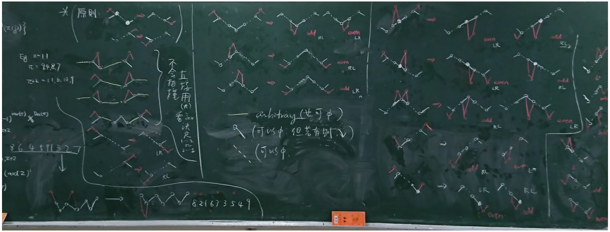
- If collisions occur, discuss case by case (more than 20 cases!).

Example



$864591327 \mapsto 821673549$





Outline

An extended result of insertion lemma:

Theorem (Eu, Fu, Hsu, Liao, Sun[3], 2019)

For $k \geq 1$ and any permutation W of $\{2k+1, 2k+2, \ldots, n\}$, we have

$$\begin{aligned}
&\sum_{\sigma \in \mathfrak{S}_n(W)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} \\
&\quad = (-1)^{\mathsf{inv}(W)} q^{\mathsf{maj}(W)} \sum_{\sigma \in \mathfrak{S}_n(2k+1:n)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} \\
&\quad = (-1)^{\mathsf{inv}(W)} q^{\mathsf{maj}(W)} [n-2k+1]_{(-1)^{n-2k}q} \cdots [n]_{(-1)^{n-1}q}.
\end{aligned}$$

Set of Subsequence restrictions

Let $\mathcal{W}$ be a set of words on $\{1, 2, \ldots, n\}$, and define

$$\mathfrak{S}_n(\mathcal{W}) = \bigcap_{W \in \mathcal{W}} \mathfrak{S}_n(W).$$

Example

$\mathfrak{S}_n(\{123, 423\}) = \{\sigma \in \mathfrak{S}_n : 123,\ 423$ are subsequences of $\sigma\}$.

Theorem (Eu, Fu, Hsu, Liao, Sun[3], 2019)

For any set $\mathcal{W}$ of words on $\{1, 2, \ldots, n-2\}$, there is a bijection $\Phi : \mathfrak{S}_n(\mathcal{W}) \to \mathfrak{S}_n(\mathcal{W} + 2)$ such that

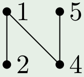
$$\mathrm{Des}(\Phi(\sigma)) = \mathrm{Des}(\sigma) \;\; \textit{and} \;\; \mathrm{inv}(\Phi(\sigma)) \equiv \mathrm{inv}(\sigma) \pmod 2$$

Restrictions by Labelings of Posets

Given a poset $(P, \preceq)$ and an injective labeling of $(P, \preceq)$: $f : P \to \{1, 2, \ldots, n\}$, define

$$\mathfrak{S}_n(f) = \{\sigma \in \mathfrak{S}_n : \sigma^{-1}(f(x)) < \sigma^{-1}(f(y)) \text{ if } x \preceq y \text{ in } P\}$$

Example

$f :$ 

$\mathfrak{S}_n(f) = \{\sigma \in \mathfrak{S}_n : 21,\ 41,\ 45$ are subsequences of $\sigma\}$.

Theorem (Eu, Fu, Hsu, Liao, Sun[3], 2019)

For any poset $(P, \preceq)$ with $|P| \leq n-2$ and any injective labeling $f : P \to \{1, 2, \ldots, n-2\}$, we have

$$\sum_{\sigma \in \mathfrak{S}_n(f)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n(f+2)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)}$$

Restrictions by Pattern Avoidance

Given $S \subseteq \{1, 2, \ldots, n\}$ and a pattern π, let

$$\mathfrak{S}_n(\pi; S) = \{\sigma \in \mathfrak{S}_n : \sigma \text{ contains no } \pi\text{-pattern when restricted to } S\}.$$

Example

$\mathfrak{S}_5(12; \{3, 4, 5\}) = \{\sigma \in \mathfrak{S}_5 : 34,\ 35,\ 45$ are not subsequences of $\sigma\}$.

Theorem (Eu, Fu, Hsu, Liao, Sun[3], 2019)

For a pattern π and a set $S \subseteq \{1, 2, \ldots, n-2\}$, we have

$$\sum_{\sigma \in \mathfrak{S}_n(\pi; S)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n(\pi; S+2)} (-1)^{\mathsf{inv}(\sigma)} q^{\mathsf{maj}(\sigma)}.$$

Outline

- We work on the signed major distributions on permutations with subsequence restrictions.
- Permutations form the Coxeter groups of type A, there are Coxeter groups of other types with combinatorial models (e.g. types B and D).
- Adin, Gessel, Roichman[1] obtained the following type B analogue

$$\sum_{\sigma\in B_n} \mathsf{sign}(\sigma) q^{\mathsf{fmaj}(\sigma)} = [2]_{-q}[4]_q \cdots [2n]_{(-1)^n q}.$$

- Results in Coxeter groups of types B and D?

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Thank you for your attention!