Parity Considerations in Rogers–Ramanujan–Gordon Type Overpartitions

Diane Y.H. Shi
joint work with Doris D.M. Sang and Ae Ja Yee

Tianjin University

shiyahui@tju.edu.cn

Taiwan, August 22, 2019

Overview

1. Background of parity considerations in integer partition
2. Background of Rogers–Ramanujan–Gordon type overpartition
3. Parity considerations in Rogers–Ramanujan–Gordon type overpartitions
4. Multisum generating functions

Background of parity considerations in partition

Definitions

Definition 1.1

A partition λ *of a positive integer* n *is a finite nonincreasing sequence of positive integers* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, *such that* $\sum_{i=1}^{r} \lambda_i = n$. λ_i *are called the parts of* λ. *The partition of* 0 *is empty.*

Definitions

Definition 1.1

A partition λ of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, such that $\sum_{i=1}^{r} \lambda_i = n$. λ_i are called the parts of λ. The partition of 0 is empty.

For $n = 4$, there are 5 partitions of 4, that is,

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

$$(4), (3,1), (2,2), (2,1,1) \quad \text{and} \quad (1,1,1,1).$$

Here and throughout this paper, we adopt the common notation defined as follows.
Let

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

We also write

$$(a_1,\ldots,a_k;q)_\infty = (a_1;q)_\infty \cdots (a_k;q)_\infty.$$

Generating function of $p(n)$

We derive Euler's generating function for the sequence $\{p(n)\}_{n=0}^{\infty}$,

Generating function of $p(n)$

We derive Euler's generating function for the sequence $\{p(n)\}_{n=0}^{\infty}$,

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{(q;q)_\infty}$$
$$= (1 + q^1 + q^{1+1} + q^{1+1+1} + \cdots)$$
$$\times (1 + q^2 + q^{2+2} + q^{2+2+2} + \cdots)$$
$$\times (1 + q^3 + q^{3+3} + q^{3+3+3} + \cdots)$$
$$\times \; etc..$$

$q^{1+1+1} \times q^2 \times q^{3+3}$ generates partition $(3,3,2,1,1,1)$.

Parity in partition

In partition theory, parity always plays a role, for example, Euler's partition theorem and the first Göllnitz-Gordon theorem.

Theorem 1 (Euler's partition identity)

The number of partitions of any positive integer n into distinct parts equals the number of partitions of n into odd parts.

Theorem 2 (First Göllnitz-Gordon identity)

The number of partitions of n into distinct nonconsecutive parts with no even parts differing by exactly 2 equals the number of partitions of n into parts $\equiv 1,\ 4,$ or $7 \pmod 8$.

The Rogers–Ramanujan identities were found independently by Rogers 1894, Ramanujan 1913 and Schur 1917.

Theorem 3 (Rogers 1894, Ramanujan 1913, Schur 1917)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{2}$$

Combinatorial version of the Rogers–Ramanujan identities

The identity (1) can be combinatorial interpreted as the following theorem:

Theorem 4 (Macmahon, 1918)

The number of partitions of n into parts differing at least 2 equals the number of partitions of n into parts which are congruent to 1 or 4 modulo 5.

The identity (2) can be combinatorial interpreted as the following theorem:

Theorem 5

The number of partitions of n into parts differing at least 2 and part 1 do not appear equals the number of partitions of n into parts which are congruent to 2 or 3 modulo 5.

Combinatorial generalization of the Rogers–Ramaujan identities

Theorem 6 (Gordon, Amer. J. Math.,1961)

Let $B_{k,i}(n)$ *denote the number of partitions of* n *of the form* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$, *where* $\lambda_j - \lambda_{j+k-1} \geq 2$ *and at most* $i-1$ *of the* λ_j *are equal to* 1 *and* $1 \leq i \leq k$. *Let* $A_{k,i}(n)$ *denote the number of partitions of* n *into parts* $\not\equiv 0, \pm i \pmod{2k+1}$. *Then for all* $n \geq 0$,

$$A_{k,i}(n) = B_{k,i}(n).$$

B. Gordon, A combinatorial generalization of the Rogers–Ramanujan identities, Amer. J. Math. 83 (1961) 393–399.

Analytic generalization of the Rogers–Ramaujan identities

Theorem 7 (Andrews, Proc. Natl. Acad. Sci. USA, 1974)

For $k \geq i \geq 1$, we have

$$\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty},$$

where we assume that $N_k = 0$.

G. E. Andrews, An analytic generalization of the Rogers–Ramanujan identities for odd moduli, Proc. Nat. Acad. Sci. USA 71 (1974) 4082–4085.

Parity considerations in Rogers–Ramanujan–Gordon type partitions

Theorem 8 (Andrews, Ramanujan J., 2010)

Suppose $k \geq a \geq 1$ are integers with $k \equiv a \pmod 2$. Let $W_{k,a}(n)$ denote the number of those partitions enumerated by $B_{k,a}(n)$ with the added restriction that even parts appear an even number of times. Then we have

$$\sum_{n\geq 0} W_{k,a}(n)q^n = \frac{(-q;q^2)_\infty (q^a, q^{2k+2-a}, q^{2k+2}; q^{2k+2})_\infty}{(q^2;q^2)_\infty}$$

$$= \sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+2N_a+2N_{a+2}+\cdots+2N_{k-2}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}} \quad (3)$$

G. E. Andrews, Parity in partition identities, Ramanujan J. 23 (2010) 45–90.

Parity considerations in Rogers–Ramanujan–Gordon type partitions

Theorem 9 (Andrews, Ramanujan J., 2010)

Suppose $k \geq a \geq 1$ with k odd and a even. Let $\overline{W}_{k,a}(n)$ denote the number of those partitions enumerated by $B_{k,a}(n)$ with added restriction that odd parts appear an even number of times. Then

$$\sum_{n\geq 0} \overline{W}_{k,a}(n)q^n = \frac{(-q^2;q^2)_\infty (q^a, q^{2k+2-a}, q^{2k+2}; q^{2k+2})_\infty}{(q^2;q^2)_\infty}$$

$$= \sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+n_1+n_3+\cdots+n_{a-3}+N_{a-1}+N_a+\cdots+N_{k-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}} \quad (4)$$

where $n_i = N_i - N_{i+1}$.

For k and a have different parities, Kursungöz discovered the following generating function by using combinatorial way.

$$\sum_{n\geq 0} W_{k,a}(n)q^n = \sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+2N_a+2N_{a+2}+\cdots+2N_{k-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}$$

Theorem 10 (Kim and Yee, JCTA 2013)

For $k \geq a \geq 1$ and $k \not\equiv a \pmod 2$,

$$\sum_{n\geq 0} W_{k,a}(n)q^n = \frac{(-q^3;q^2)_\infty(q^{a+1},q^{2k+1-a},q^{2k+2};q^{2k+2})_\infty}{(q^2;q^2)_\infty} + \frac{q(-q^3;q^2)_\infty(q^{a-1},q^{2k+3-a},q^{2k+2};q^{2k+2})_\infty}{(q^2;q^2)_\infty}. \qquad (5)$$

K. Kurşungöz, Parity considerations in Andrews-Gordon identities, and the k-marked Durfee symbols, PhD thesis, Penn Sate University, 2009.
S. Kim and A.J. Yee, Rogers-Ramanujan-Gordon identities, generalized Göllnitz-Gordon identities, and parity questions. J. Comb. Theory, Ser. A 120(5), 1038–1056 (2013).

Parity considerations in Rogers–Ramanujan–Gordon type partitions

For $\overline{W}_{k,a}(n)$, Kim and Yee showed the following result which is a "missing" case of Andrews.

Theorem 11 (Kim and Yee, JCTA 2013)

Suppose $k \geq a \geq 1$ with k even and a odd. Then

$$\sum_{n\geq 0} \overline{W}_{k,a}(n)q^n = \frac{(-q^2;q^2)_\infty (q^{a+1}, q^{2k+1-a}, q^{2k+1}; q^{2k+1})_\infty}{(q^2;q^2)_\infty} \tag{6}$$

$$= \sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+n_1+n_3+\cdots+n_{a-3}+N_{a-1}+N_a+\cdots+N_{k-1}}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}. \tag{7}$$

Parity considerations in Rogers–Ramanujan–Gordon type overpartitions

They also noted the following relation for $\overline{W}_{k,a}(n)$.

Theorem 12 (Kim and Yee, JCTA 2013)

For $k \geq a \geq 1$ with a even and $n \geq 1$.

$$\overline{W}_{k,a}(n) = \overline{W}_{k,a-1}(n). \tag{8}$$

Conjecture of Andrews

They also noted the following relation for $\overline{W}_{k,a}(n)$.

Conjecture 1.2 (Andrews, Ramanujan J., 2010)

Extend the parity indices to overpartitions in a manner that will provide natural generalizations of the work of Corteel and Lovejoy.

Background of Rogers–Ramanujan–Gordon type overpartition

The Definition of Overpartition

Definition 2.1

An overpartition λ *of a positive integer* n *is also a non-increasing sequence of positive integers* $\lambda_1 \geq \cdots \geq \lambda_s > 0$ *such that* $n = \lambda_1 + \cdots + \lambda_s$ *and the first occurrence of each integer may be overlined. The overpartition of zero is the overpartition with no parts.*

The Definition of Overpartition

Definition 2.1

An overpartition λ of a positive integer n is also a non-increasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_s > 0$ such that $n = \lambda_1 + \cdots + \lambda_s$ and the first occurrence of each integer may be overlined. The overpartition of zero is the overpartition with no parts.

For example, $(\overline{7}, 7, 6, \overline{5}, 2, \overline{1})$ is an overpartition of 28.

The Definition of Overpartition

Definition 2.1

An overpartition λ of a positive integer n is also a non-increasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_s > 0$ such that $n = \lambda_1 + \cdots + \lambda_s$ and the first occurrence of each integer may be overlined. The overpartition of zero is the overpartition with no parts.

For example, $(\overline{7}, 7, 6, \overline{5}, 2, \overline{1})$ is an overpartition of 28.
For an overpartition λ and for any integer l, let $f_l(\lambda)(f_{\overline{l}}(\lambda))$ denote the number of occurrences of l non-overlined (overlined) in λ.

Let $\overline{p}(n)$ denotes the number of overpartitions of n. Then the generating function of $\overline{p}(n)$ is the

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{9}$$

The overpartition analogue of the Rogers–Ramanujan–Gordon theorem in two special cases were given by Lovejoy in 2003 and then the theorem for all cases was given by Chen, Sang and Shi in 2013.

Overpartition analogue of the Rogers–Ramanujan–Gordon Theorem

Theorem 13 (Chen, Sang and S., Proc. London Math. Soc. 2013)

For $k \geq i \geq 1$, let $\overline{B}_{k,i}(n)$ denote the number of overpartitions of n of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$, such that 1 can occur as a non-overlined part at most $i-1$ times, and where $\lambda_j - \lambda_{j+k-1} \geq 1$ if λ_j is overlined and $\lambda_j - \lambda_{j+k-1} \geq 2$ otherwise. For $k > i \geq 1$, let $\overline{A}_{k,i}(n)$ denote the number of overpartitions of n whose non-overlined parts are not congruent to $0, \pm i$ modulo $2k$ and let $\overline{A}_{k,k}(n)$ denote the number of overpartitions of n with parts not divisible by k. Then $\overline{A}_{k,i}(n) = \overline{B}_{k,i}(n)$.

W.Y.C. Chen, D.D.M. Sang and D.Y.H. Shi, the Rogers-Ramanujan -Gordon theorem for overpartitions, Proc. London Math. Soc. **106***(3) (2013) 1371–1393.*

J. Lovejoy, Gordon's theorem for overpartitions, J. Combin. Theory, Ser. A. **103** *(2003) 393–401.*

Overpartition analogue of the Andrews–Gordon type identities

Theorem 14 (Chen, Sang and S., Proc. London Math. Soc. 2013)

$$\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}(1+q^{N_i})}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}$$
$$=\frac{(-q)_\infty(q^i,q^{2k-i},q^{2k};q^{2k})_\infty}{(q)_\infty}. \tag{10}$$

Parity considerations in Rogers–Ramanujan–Gordon type overpartitions

Parity considerations in Rogers–Ramanujan–Gordon type overpartitions

Now we consider the parity question in Rogers–Ramanujan–Gordon type overpartitions.

The partitions enumerated by $W_{k,a}(n)$ and $\overline{W}_{k,a}(n)$ are defined by putting parity restrictions in the partitions enumerated by $B_{k,a}(n)$. Here instead of adding extra parity restrictions directly to Rogers–Ramanujan–Gordon type overpartitions enumerated by $\overline{B}_{k,a}(n)$, we shall define two new Rogers–Ramanujan-Gordon type overpartitions with parity considerations as follows.

Parity considerations in Rogers–Ramanujan–Gordon type overpartitions

Definition 15 (Sang, S. and Yee, preprint)

For $k \geq a \geq 1$, let $U_{k,a}(n)$ denote the number of overpartitions of n of the form $(\overline{1}^{f_{\overline{1}}}, 1^{f_1}, \overline{2}^{f_{\overline{2}}}, 2^{f_2}, \ldots)$ such that

(i) $f_1(\lambda) \leq a - 1 + f_{\overline{1}}(\lambda)$;

(ii) $f_{2l-1}(\lambda) \geq f_{\overline{2l-1}}(\lambda)$;

(iii) $f_{2l}(\lambda) + f_{\overline{2l}}(\lambda) \equiv 0 \pmod 2$;

(iv) $f_l(\lambda) + f_{\overline{l}}(\lambda) + f_{l+1}(\lambda) \leq k - 1 + f_{\overline{l+1}}(\lambda)$.

Parity considerations in Rogers–Ramanujan–Gordon type overpartitions

Definition 3.1 (Sang, S. and Yee, preprint)

For $k \geq a \geq 1$, let $\overline{U}_{k,a}(n)$ denote the number of overpartitions of n of the form $(\overline{1}f_{\overline{1}},\ 1f_1,\ \overline{2}f_{\overline{2}},\ 2f_2, \ldots)$ such that

(i) $f_1(\lambda) \leq a - 1 + f_{\overline{1}}(\lambda)$;

(ii) $f_{2l}(\lambda) \geq f_{\overline{2l}}(\lambda)$;

(iii) $f_{2l-1}(\lambda) + f_{\overline{2l-1}}(\lambda) \equiv 0 \pmod 2$;

(iv) $f_l(\lambda) + f_{\overline{l}}(\lambda) + f_{l+1}(\lambda) \leq k - 1 + f_{\overline{l+1}}(\lambda)$.

Parity considerations in Rogers–Ramanujan–Gordon type overpartitions

Theorem 16 (Sang, S. and Yee, preprint)

For $k \geq a \geq 1$ and $k \equiv a \pmod 2$, we have

$$\sum_{n\geq 0} U_{k,a}(n)q^n = \frac{(-q;q)_\infty (q^a, q^{2k-a}, q^{2k}; q^{2k})_\infty}{(q^2;q^2)_\infty}. \tag{11}$$

Theorem 17 (Sang, S. and Yee, preprint)

For $k \geq a \geq 1$ and $k \not\equiv a \pmod 2$, we have

$$\sum_{n\geq 0} U_{k,a}(n)q^n = \frac{(-q^2;q)_\infty (q^{a+1}, q^{2k-a-1}, q^{2k}; q^{2k})_\infty}{(q^2;q^2)_\infty} + \frac{q(-q^2;q)_\infty (q^{a-1}, q^{2k-a+1}, q^{2k}; q^{2k})_\infty}{(q^2;q^2)_\infty}. \tag{12}$$

Parity considerations in Rogers–Ramanujan–Gordon type overpartitions

Theorem 18 (Sang, S. and Yee, preprint)

For $k \geq a \geq 2$ with a even, we have

$$\sum_{n\geq 0} \overline{U}_{k,a}(n)q^n = \sum_{n\geq 0} \overline{U}_{k,a-1}(n)q^n = \frac{(-q^2;q^2)_\infty^2 (q^a, q^{2k-a}, q^{2k}; q^{2k})_\infty}{(q^2;q^2)_\infty}. \tag{13}$$

Parity considerations in Rogers–Ramanujan–Gordon type overpartitions

$$\overline{Q}_{k,a}(x;q) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{kn} q^{kn^2+kn-an}(1-x^a q^{(2n+1)a})(-xq^{n+1})_\infty(-q)_n}{(q)_n(xq^{n+1})_\infty}, \tag{14}$$

$$\overline{Q}_{k,a}(x;q) = H_{k,a}(-1/q;xq;q)$$

G.E. Andrews, The Theory of Partitions, Addison-Wesley, Reading, MA, 1976; reissued by Cambridge University Press, Cambridge, 1998.

The generating function of $U_{2k,2a}(n)$

Theorem 19 (Sang, S. and Yee, preprint)

For $k \geq a \geq 1$ are integers, we have

$$U_{2k,2a}(x;q) = (-xq;q^2)_\infty \overline{Q}_{k,a}(x^2;q^2). \tag{15}$$

The generating function of $\overline{U}_{2k,2a}(n)$

Theorem 20 (Sang, S. and Yee, preprint)

For $k \geq a \geq 2$ with a even, we have

$$\overline{U}_{k,a}(n) = \overline{U}_{k,a-1}(n). \tag{16}$$

Theorem 21 (Sang, S. and Yee, preprint)

For $k \geq a \geq 1$, we have

$$\overline{U}_{2k,2a}(x;q) = (-xq^2;q^2)_\infty \overline{Q}_{k,a}(x^2;q^2).$$

The generating function of $U_{2k+1,2a+1}(n)$ and $\overline{U}_{2k+1,2a+1}(n)$

Theorem 22 (Sang, S. and Yee, preprint)

For $k \geq a \geq 0$, we have

$$U_{2k+1,2a+1}(x;q) = (-xq;q^2)_\infty \overline{Q}_{k+\frac{1}{2},a+\frac{1}{2}}(x^2;q^2). \tag{17}$$

$$\overline{U}_{2k+1,2a}(x;q) = \overline{U}_{2k+1,2a-1}(x;q) = (-xq^2;q^2)_\infty \overline{Q}_{k+\frac{1}{2},a}(x^2;q^2). \tag{18}$$

$$\sum_{n=0}^{\infty} U_{2k+1,2a+1}(n)q^n = \frac{(-q;q)_\infty (q^{2a+1}, q^{4k+1-2a}, q^{4k+2}; q^{4k+2})_\infty}{(q^2;q^2)_\infty}.$$

$$\sum_{n=0}^{\infty} \overline{U}_{2k+1,2a}(n)q^n = \sum_{n=0}^{\infty} \overline{U}_{2k+1,2a-1}(n)q^n$$
$$= \frac{(-q^2;q^2)_\infty^2 (q^{2a}, q^{4k-2a+2}, q^{4k+2}; q^{4k+2})_\infty}{(q^2;q^2)_\infty}.$$

The generating function of $U_{2k,2a-1}(n)$ and $U_{2k+1,2a}(n)$

Theorem 23 (Sang, S. and Yee, preprint)

For $k \geq a \geq 1$, we have

$$U_{2k,2a}(x;q) - U_{2k,2a-1}(x;q) = xq\left[U_{2k,2a-1}(x;q) - U_{2k,2a-2}(x;q)\right]. \quad (19)$$

$$U_{2k,2a-1}(x;q) = \frac{1}{1+xq}U_{2k,2a}(x;q) + \frac{xq}{1+xq}U_{2k,2a-2}(x;q).$$

$$\sum_{n\geq 0} U_{2k,2a-1}(n)q^n = \frac{(-q^2;q)_\infty (q^{2a}, q^{4k-2a}, q^{4k}; q^{4k})_\infty}{(q^2;q^2)_\infty} + \frac{q(-q^2;q)_\infty (q^{2a-2}, q^{4k-2a+2}, q^{4k}; q^{4k})_\infty}{(q^2;q^2)_\infty} \quad (20)$$

Multisum generating functions

Multisum generating functions

Theorem 24 (Sang, S. and Yee, preprint)

For $k \equiv a \pmod 2$, we have

$$\sum_{n\geq 0} U_{k,a}(n)q^n$$
$$= \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_1^2+N_3^2+N_4^2+\cdots+N_{2k-1}^2+N_2+2N_{2a+2}+\cdots+2N_{2k-2}}(-q^2;q^2)_{N_2-1}(1+q^{2N_{2a}})}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{2k-2}}(q^2;q^2)_{n_{2k-1}}}$$

Multisum generating functions

Theorem 25 (Sang, S. and Yee, preprint)

$$\sum_{n\geq 0}\overline{U}_{2k,2a}(n)q^n$$
$$=\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{N_1^2+N_3^2+N_4^2+\cdots+N_{2k-1}^2+N_2+2N_{2a+2}+\cdots+2N_{2k-2}+n_1+n_3+\cdots+n_{2k-1}}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{2k-2}}(q^2;q^2)_{n_{2k-1}}}$$
$$\times(-q^2;q^2)_{N_2-1}(1+q^{2N_{2a}})$$

Multisum generating functions

Theorem 26 (Sang, S. and Yee, preprint)

For $k \geq a \geq 1$, we have

$$\sum_{n\geq 0} \overline{U}_{2k-1,2a}(n)q^n$$

$$= \sum_{N_1\geq\cdots\geq N_{2k-1}\geq 0} \frac{q^{N_1^2+N_2+\sum_{i=3}^{2k-1} N_i^2+2\sum_{i=a+1}^{k-1} N_{2i}+\sum_{i=1}^{k} n_{2i-1}}(-q^2;q^2)_{N_2-1}(1+q^{2N_{2a}})}{\prod_{i=1}^{2k-1}(q^2;q^2)_{N_i-N_{i+1}}(-q^2;q^2)_{N_{2k-2}}}, \tag{21}$$

where $n_i = N_i - N_{i-1}$.

Thanks for attending!