Combinatorial Aspects of Infinitesimal Free Probability with Amalgamation

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Question

Assume that we know the infinitesimal distribution of $x = x^*$, $y = y^*$.
Given an self-adjoint polynomial P with variables x & y. Whether we can write down the precise formula for the infinitesimal distribution of P ?

Linearization Trick

$(\mathcal{A}, \varphi, \varphi')$ C^* infinitesimal Prob space.
$x_1, \dots, x_n$ are self-adjoint where are infinitesimally free. $P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$ is self-adjoint

step 1.

$$P = P(x_1, \dots, x_n)$$

$$\downarrow$$

$$\mathcal{L}_P = b_0 \otimes 1 + b_1 \otimes x_1 + \dots + b_n \otimes x_n$$

where b_i is self-adjoint

$\Longrightarrow$

step 2.

$x_1, x_2, \dots, x_n$ are infinitesimally free

$$\downarrow$$

$b_1 \otimes x_1, \dots, b_n \otimes x_n$ are infinitesimally free w.r.t $(M_N(\mathcal{A}), \varphi \otimes Id_N, \varphi' \otimes Id_N)$

$\Longrightarrow$

step 3.

$$g_{\mathcal{L}_P}(b) = g_{\mathcal{L}_P - b_0 \otimes 1}(b - b_0)$$

$$= g_{b_1 \otimes x_1 + \dots + b_n \otimes x_n}(b - b_0)$$

$$\downarrow$$

$$g_P(z) = \lim_{\varepsilon \downarrow 0} \left[g_{\mathcal{L}_P}\left(\begin{bmatrix} z & & \\ & i\varepsilon & \\ & & \ddots & \\ & & & i\varepsilon \end{bmatrix} \right) \right]_{1,1}$$

$$\downarrow$$

Stieltjes Inversion Formula.

Free Probability

Classical Probability	Free Probability
$(\Omega, \mathcal{F}, P)$: Prob space.	$(\mathcal{A}, \varphi)$: non-commutative Probability space
$L^{\infty-}(\Omega, \mathcal{F}, P) = \{$ random variables have all moments $\}$.	if $\mathcal{A}$ is a unital algebra
$E(X) = \int_\Omega X \, dP$. $\quad (E(1) = 1)$	$\varphi : \mathcal{A} \to \mathbb{C}$ is linear with $\varphi(1) = 1$
μ_X : distribution of X : $\mu_X(A) = P(X \in A)$	Given $a \in \mathcal{A}$, the distribution of a is the sequence $\{\varphi(a^k) \mid k \geq 1\}$.
$E(X^k) = \int t^k \, d\mu_X(t)$: the k-th moment	
X, Y are (classically) independent	$a, b \in \mathcal{A}$ are freely independent
iff $E[f(X) g(Y)] = E[f(X)] E[g(Y)]$	if ????

Free Probability

Given a non-comm Prob space $(\mathcal{A}, \varphi)$, $\mathcal{A}_1$ & $\mathcal{A}_2 \subset \mathcal{A}$ are unital subalgebras.

- $\mathcal{A}_1$ & $\mathcal{A}_2$ are <u>free</u> (or <u>freely independent</u>) if $\forall\, n \in \mathbb{N}$, $j = 1, 2, \dots, n$.

  $i(j) \in \{1, 2\}$, $i(1) \neq i(2) \neq \cdots \neq i(n)$, $a_j \in \mathcal{A}_{i(j)}$ & $\varphi(a_j) = 0 \Rightarrow \varphi(a_1 a_2 \cdots a_n) = 0$

- $a, b \in \mathcal{A}$ are <u>free</u> if $\mathcal{A}_1 = \mathrm{alg}(a, 1)$ & $\mathcal{A}_2 = \mathrm{alg}(b, 1)$ are free.

Notation : $[n] = \{1, 2, 3, \dots, n\}$ & $\pi \in NC(n)$: non-crossing partition of $[n]$

(i.e, $\nexists\, p_1 < p_2 < q_1 < q_2$ such that $p_1, q_1 \in V_i$ &
$p_2, q_2 \in V_j$.

where $V_i, V_j \in \pi$, $i \neq j$)

- <u>Free Cumulants</u> $k_n : \mathcal{A}^n \to \mathbb{C}$ is the multilinear functional defined recursively by $\varphi(a_1 a_2 \cdots a_n) = \sum_{\pi \in NC(n)} k_\pi[a_1, a_2, \dots, a_n]$, where $k_\pi[a_1, \dots, a_n] = \prod_{\pi\ \{V_1, \dots V_r\}} k_{|V_j|}[a_{i_1}, \dots, a_{i_s}]$

  $V_j = \{a_{i_1}, \dots, a_{i_s}\}$

<u>Theorem</u> (<u>Vanishing of Mixed Cumulants</u>)

$\mathcal{A}_1, \mathcal{A}_2$ are free $\Longleftrightarrow$ $k_n[a_1, \dots, a_n] = 0$ whenever $\exists\, k, l \in [n]$ such that $a_k \in \mathcal{A}_{i_k}$ & $a_l \in \mathcal{A}_{i_l}$ & $i_k \neq i_l$

Links with Random Matrices

- $a_{ij} \overset{\text{iid}}{\sim} N(0,1)$ for all $i,j = 1,2,\dots,N$ & $G = [a_{ij}]_{i,j=1}^{N}$.

  Let $A_N = \frac{1}{\sqrt{2N}}(G + G^t)$. Then A_N is called the Gaussian Orthogonal Ensemble (GOE)

- Let $\mathcal{A}_N = M_N(L^{\infty-}(\Omega, \mathcal{F}, P))$ & $\varphi_N = \frac{1}{N} E \circ Tr(\cdot)$.

  (Wigner). $A_N \in \mathcal{A}_N$. $\lim\limits_{N \to \infty} \varphi_N(A_N^k) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ C_\ell & \text{if } k \text{ is even. } k = 2\ell \end{cases}$

  $$= \int_{-2}^{2} t^k \underbrace{\frac{1}{2\pi}\sqrt{4-t^2}\, dt}_{\text{semi circle law}} = \varphi(s^k)$$

  s : semicircular element in $(\mathcal{A}, \varphi)$

Theorem (Voiculescus, 1991)

$\{A_N\}_N$, $\{B_N\}_N$ GOE + $\{a_{ij}^{(N)}\}_{i,j=1}^{N}$, $\{b_{ij}^{(N)}\}_{i,j=1}^{N}$ classically independent $\Rightarrow$ $\exists$ semicircular elements s_1 & s_1 in a ncps $(\mathcal{A}, \varphi)$ such that $\varphi_N(p(A_N, B_N)) \to \varphi(p(s_1, s_2))$ & s_1, s_2 are free.

$\underbrace{\qquad\qquad\qquad}_{\text{asymptotically free.}}$

Infinitesimal Freeness

- A_N : GOE. $\varphi_N(A_N^k) \longrightarrow \varphi(s^k)$ as $N \to \infty$

$$\varphi_N(A_N^k) = \varphi(s^k) + \frac{1}{N}\Box + o\left(\frac{1}{N}\right) \Longrightarrow \Box = ???$$

(Mingo. 2018 ; N. Enriquez & L. Menard 2016)

$$\Box = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{1}{2}\left[2^k - \binom{k}{k/2}\right] & \text{if } k \text{ is even} \end{cases} = \underbrace{\frac{1}{2}\left\{\frac{1}{2}(\delta_{-2}+\delta_2) - \frac{\chi_{[-2,2]}}{\pi\sqrt{4-t^2}}\,dt\right\}}_{\frac{1}{2}(\text{Bernoulli} - \text{Arcsine})}$$

- $(\mathcal{A}, \varphi, \varphi')$: infinitesimal non-commutative Probability space (incps)

  if $\mathcal{A}$ is a unital algebra & $\varphi, \varphi' : \mathcal{A} \to \mathbb{C}$ are Linear functional s.t. $\varphi(1)=1$, $\varphi'(1)=0$.

- $\mathcal{A}_1, \mathcal{A}_2 \subseteq (\mathcal{A}, \varphi, \varphi')$ two unital subalgebras are infinitesimally free if for $n \in \mathbb{N}$

for $a_1, a_2, \dots, a_n \in \mathcal{A}$ such that $a_k \in \mathcal{A}_{i(k)}$, $i(1) \neq i(2) \neq \cdots \neq i(n)$. $\varphi(a_1) = \cdots = \varphi(a_n) = 0$

then $\varphi(a_1 a_2 \cdots a_n) = 0$;

$$\varphi'(a_1 a_2 \cdots a_n) = \sum_{j=1}^{n} \varphi(a_1 \cdots a_{j-1}\, \varphi'(a_j)\, a_{j+1} \cdots a_n).$$

Infinitesimal Freeness

- Infinitesimal cumulants $\kappa_n' : \mathcal{A}^n \to \mathbb{C}$ is the multilinear functional define by the following relation : $\varphi'(a_1 a_2 \cdots a_n) = \sum_{\pi \in NC(n)} \partial\kappa_\pi[a_1, \ldots, a_n]$

where $\partial\kappa_\pi[a_1, \ldots, a_n] = \sum_{V \in \pi} \kappa'_{\pi, V}(a_1, \ldots, a_n) = \sum_{\substack{V \in \pi \\ V = (i_1, \ldots, i_\ell)}} \kappa'_{|V|}[a_{i_1}, \ldots, a_{i_\ell}] \prod_{\substack{W \in \pi \\ W \neq V \\ W = (j_1, \ldots, j_k)}} \kappa_{|W|}[a_{j_1}, \ldots, a_{j_k}]$

Ex. $\pi = \{(1,4), (2,3), (5)\} \in NC(5)$.

$$\begin{aligned} \partial\kappa_\pi[a_1, a_2, a_3, a_4, a_5] &= \kappa'_{|(1,4)|}(a_1, a_4)\, \kappa_{|(2,3)|}(a_2, a_3)\, \kappa_{|(5)|}(a_5) + \kappa_{|(1,4)|}(a_1, a_4)\, \kappa'_{|(2,3)|}(a_2, a_3)\, \kappa_{|(5)|}(a_5) \\ &+ \kappa_{|(1,4)|}(a_1, a_4)\, \kappa_{|(2,3)|}(a_2, a_3)\, \kappa'_{|(5)|}(a_5) \\ &= \kappa_2'(a_1, a_4)\, \kappa_2(a_2, a_3)\, \kappa_1(a_5) + \kappa_2(a_1, a_4)\, \kappa_2'(a_2, a_3)\, \kappa_1(a_5) + \kappa_2(a_1, a_4)\, \kappa_2(a_2, a_3)\, \kappa_1'(a_5) \end{aligned}$$

- (Fevrier & Nica 2010)

$\mathcal{A}_1, \mathcal{A}_2$ are infinitesimally free $\iff$ $\kappa_n[a_1, \ldots, a_n] = \kappa_n'[a_1, \ldots, a_n] = 0$ whenever $\exists\, k, \ell \in [n]$ s.t. $a_k \in \mathcal{A}_{i(k)}$ & $a_\ell \in \mathcal{A}_{i(\ell)}$.

- (Shlyakhtenko, 2015)

A_N : GOE & $B_N = \begin{bmatrix} \theta_1 & & & & & \\ & \ddots & & & & \\ & & \theta_{N_0} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}_{N \times N}$ $\Rightarrow$ $\left\{ \begin{array}{l} \varphi_N[p(A_N, B_N)] \longrightarrow \varphi(p(s,t)) \\ N[\varphi_N(p(A_N, B_N)) - \varphi(p(s,t))] \longrightarrow \varphi'(p(s,t)) \end{array} \right.$ & s, t are inf. free $\}$

asymptotically infinitesimally free

Operator-Valued Free Probability

Scalar Version:

$\mathcal{A}$: unital algebra.

$\varphi : \mathcal{A} \to \mathbb{C}$ is linear with $\varphi(1) = 1$

$(\mathcal{A}, \varphi)$ is a non-commutative Prob space (ncps)

$\Longrightarrow$

Operator Version:

$\mathcal{M} = M_N(\mathcal{A})$; $\mathcal{B} = M_N(\mathbb{C})$; $\mathbb{E} = \varphi \otimes Id_N$

$\mathbb{E} : \mathcal{M} \to \mathcal{B}$ is linear s.t $\mathbb{E}(1) = 1$

$(\mathcal{M}, \mathbb{E}, \mathcal{B})$ is a ncps with amalgamation (or operator-valued Prob space)

- $(\mathcal{M}, \mathbb{E}, \mathcal{B})$ is an operator-valued probability space if $\mathcal{M}$ is a unital algebra & $\mathcal{B} \subseteq \mathcal{M}$ unital subalgebra & $\mathbb{E} : \mathcal{M} \to \mathcal{B}$ is linear such that $\mathbb{E}(1) = 1$ & $\mathbb{E}(b_1 a b_2) = b_1 \mathbb{E}(a) b_2$ for all $b_1, b_2 \in \mathcal{B}$, $a \in \mathcal{M}$.

- $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{M}$ are free with respect to $\mathbb{E}$ if for all $n \in \mathbb{N}$, $i(1) \neq i(2) \neq \cdots \neq i(n) \in \{1, 2\}$ $a_1 \in \mathcal{A}_{i(1)}, \ldots, a_n \in \mathcal{A}_{i(n)}$ with $\varphi(a_1) = \mathbb{E}(a_2) = \cdots = \mathbb{E}(a_n) = 0$, then $\mathbb{E}(a_1 a_2 \cdots a_n) = 0$.

- $\mathcal{B}$-Valued Cumulants $\kappa_n^{\mathcal{B}} : \mathcal{M}^n \to \mathcal{B}$ is defined by $\mathbb{E}(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi^{\mathcal{B}}(a_1, \ldots, a_n)$

Note: $\pi = \{(1,4), (2,3), (5)\} \in NC(5)$, $\kappa_\pi^{\mathcal{B}}(a_1, \ldots, a_5) = \kappa_2^{\mathcal{B}}[a_1 \kappa_2^{\mathcal{B}}(a_2, a_3), a_4]\, \kappa_1^{\mathcal{B}}(a_5)$

- $\mathcal{A}_1, \mathcal{A}_2$ are free w.r.t $\mathbb{E}$ $\Longleftrightarrow$ mixed $\mathcal{B}$-Valued Cumulants $\kappa_n^{\mathcal{B}}$ vanishes

Operator-Valued Infinitesimal Probability

- $(\mathcal{M}, \mathbb{E}, \mathbb{E}', \mathcal{B})$: operator-valued infinitesimal probability space if

  $\mathcal{M}$: unital algebra & $\mathcal{B} \subseteq \mathcal{M}$ unital subalgebra.

  $\mathbb{E}, \mathbb{E}' : \mathcal{M} \to \mathcal{B}$ are linear, $\mathcal{B}$-$\mathcal{B}$ bimodule with $\mathbb{E}(1) = 1$, $\mathbb{E}'(1) = 0$.

- $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{M}$ are subalgebras which containing $\mathcal{B}$, are infinitesimally free with respect to $(\mathbb{E}, \mathbb{E}')$ if for $i(1) \neq i(2) \neq \cdots \neq i(n) \in \{1, 2\}$, $a_1 \in \mathcal{A}_{i(1)}, \ldots, a_n \in \mathcal{A}_{i(n)}$

  $\mathbb{E}(a_1) = \cdots = \mathbb{E}(a_n) = 0$, then
$$\begin{cases} \mathbb{E}(a_1 a_2 \cdots a_n) = 0 \ ; \\ \mathbb{E}'(a_1 \cdots a_n) = \sum_{j=1}^{n} \mathbb{E}\left(a_1 \cdots a_{j-1} \mathbb{E}'(a_j) a_{j+1} \cdots a_n\right). \end{cases} \quad \textbf{(}\Delta\textbf{)}$$

Remark.

$$(\Delta) = \begin{cases} \mathbb{E}\left(a_1 \mathbb{E}\left(a_2 \mathbb{E}\left(a_3 \cdots \mathbb{E}\left(a_{\frac{n-1}{2}} \mathbb{E}'\left(a_{\frac{n+1}{2}}\right) a_{\frac{n+3}{2}}\right) \cdots a_{n-2}\right) a_{n-1}\right) a_n\right) & \text{if } n \text{ is odd } \& \\ & i(1) = i(n),\ i(2) = i(n-1), \ldots, i\left(\frac{n-1}{2}\right) = i\left(\frac{n+3}{2}\right) \\ 0, & \text{otherwise.} \end{cases}$$

Operator-Valued Infinitesimal Probability

Given $(\mathcal{M}, \mathbb{E}, \mathbb{E}', \mathcal{B})$: op-valued infinitesimal prob space.

Define $\tilde{\mathcal{M}} = \left\{ \begin{pmatrix} a & a' \\ 0 & a \end{pmatrix} \;\middle|\; a, a' \in \mathcal{M} \right\} \subseteq M_2(\mathcal{M})$ & $\tilde{\mathcal{B}} = \left\{ \begin{pmatrix} b & b' \\ 0 & b \end{pmatrix} \;\middle|\; b, b' \in \mathcal{B} \right\} \subseteq M_2(\mathcal{B})$

$$\tilde{\mathbb{E}} : \tilde{\mathcal{M}} \to \tilde{\mathcal{B}} \quad \text{by} \quad \tilde{\mathbb{E}} \begin{pmatrix} a & a' \\ 0 & a \end{pmatrix} = \begin{pmatrix} \mathbb{E}(a) & \mathbb{E}'(a) + \mathbb{E}(a') \\ 0 & \mathbb{E}(a) \end{pmatrix}$$

Then $(\tilde{\mathcal{M}}, \tilde{\mathbb{E}}, \tilde{\mathcal{B}})$ is an operator-valued Probability space.

Theorem.

$\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{M}$ are inf free w.r.t $(\mathbb{E}, \mathbb{E}')$ $\Longleftrightarrow$ $\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2 \subseteq \tilde{\mathcal{M}}$ are free w.r.t $\tilde{\mathbb{E}}$

$\Updownarrow$

mixed $\mathcal{B}$-valued Cumulants $\kappa_n^{\mathcal{B}}$ vanishes
inf. Cumulants $\partial\kappa_n^{\mathcal{B}}$
$\Longleftrightarrow$ mixed $\tilde{\mathcal{B}}$-valued Cumulants $\tilde{\kappa}_n^{\mathcal{B}}$ vanishes

Corollary.

$\{a_{i,j}\}_{i,j=1}^N$ & $\{b_{i,j}\}_{i,j=1}^N$ are inf free in $(\mathcal{A}, \varphi, \varphi')$ $\Longrightarrow$ $[a_{ij}], [b_{ij}]$ are inf free in $(M_N(\mathcal{A}), \varphi \otimes Id_N, \varphi' \otimes Id_N, M_N(\mathbb{C}))$