Approximation algorithms for solving the 1-line Euclidean minimum Steiner tree problem

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1 Motivations

The minimum spanning tree problem

Many polynomial-time exact algorithms [13], such as the Kruskal algorithm, the Prim algorithm, the Greedy algorithm.

The minimum Steiner tree problem

For a weighted graph $G = (V, E; w)$ equipped with a subset $X \subseteq V$, it is asked to find a subtree T of G to interconnect all vertices in X, the objective is to minimize the total length of all edges in such a subtree T among all subtrees that interconnect all vertices in X.

Such a subtree T is called as a Steiner tree of G, each vertex in X is called a terminal and each vertex that is not in X but still in such a Steiner tree is called as a Steiner vertex or Steiner point.

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Shamos and Hoey [14] addressed the Euclidean minimum spanning tree problem.

Given a set $X = \{r_1, r_2, \dots, r_n\}$ of n points in the Euclidean plane $\mathbb{R}^2$, it is asked to construct a tree to span these n points, the objective is to minimize the total length of all edges in such a spanning tree T.

Introducing a single geometric structure, called as the Voronoi diagram, which can be constructed rapidly in time $O(n \log n)$ and contains all of the relevant proximity information in only linear space, Shamos and Hoey [14] presented an exact algorithm in time $O(n \log n)$ to solve the Euclidean minimum spanning tree problem.

The Euclidean minimum Steiner tree (EMST) problem is defined as follows.

For a set $X = \{r_1, r_2, \ldots, r_n\}$ of n points in the Euclidean plane $\mathbb{R}^2$, it is asked to find a Steiner tree T interconnecting these n terminals, the objective is to minimize the total length of all edges in such a Steiner tree T among all Steiner trees to interconnect the terminals in X.

It may add some Steiner vertices or Steiner points.

Garey, Graham and Johnson [5] showed the EMST problem is NP-hard.

It is natural to use an Euclidean minimum spanning tree to approximate an Euclidean minimum Steiner tree. The Euclidean Steiner ratio is defined as the minimum upper-bound for the ratio between total lengths of an Euclidean minimum spanning tree and an Euclidean minimum Steiner tree for the same set P of n points in $\mathbb{R}^2$.

For this Euclidean Steiner ratio, Gilbert and Pollak [9] conjectured that the Euclidean Steiner ratio is $2/\sqrt{3} \approx 1.155$.

2 Models

In 1987, Georgakopoulos and Papadimitriou [12] first addressed the 1-Steiner tree problem.

Given a set $X = \{r_1, r_2, \ldots, r_n\}$ of n terminal points in the Euclidean plane $\mathbb{R}^2$, it is asked to find a new point $s \in \mathbb{R}^2$ (if any) such that the total length of the spanning tree of $X \cup \{s\}$ is minimized.

Using Euclidean plane subdivisions called oriented Dirichlet cell partitions, the authors [12] presented an exact algorithm in time $O(n^2)$ to solve the 1-Steiner tree problem.

Another related problem is the k-Steiner tree problem [2], in which the Steiner tree may contain up to k Steiner points for a given constant k, i.e.,

Given a set $X = \{r_1, r_2, \ldots, r_n\}$ of n terminal points in the Euclidean plane $\mathbb{R}^2$, it is asked to find at most new k points $s_1, \ldots, s_k \in \mathbb{R}^2$ (if any) such that the total length of the spanning tree of $X \cup \{s_1, \ldots, s_k\}$ is minimized.

In 2015, generalizing the methods mentioned-above in [12] to solve the k-Steiner tree problem, Brazil et al. [2] showed that the k-Steiner tree problem can be solved in $O(n^{2k})$ for any fixed constant k in normed plane.

In 2000, Chen and Zhang [3] considered the constrained minimum spanning tree problem that is another variation of the 1-Steiner tree problem.

Given a fixed line l and a set $X = \{r_1, r_2, \ldots, r_n\}$ of n terminal points located on same side of this line in the Euclidean plane $\mathbb{R}^2$, it is asked to find a new point s at this line l such that the total length of the spanning tree of $X \cup \{s\}$ is minimized.

And introducing a kind of partition of the line l which applies the technique of divide-and-conquer and presenting an efficient way to update a minimum spanning tree, the authors [3] designed an exact algorithm in time $O(n^2)$ to solve this constrained minimum spanning tree problem.

In 2017, Holby [8] considered a variation of the EMST problem by introducing a fixed Steiner line, whose weight is not counted in the resulting network, in addition to the Steiner points.

Given a fixed line l and a set $P = \{r_1, r_2, \ldots, r_n\}$ of n points in the Euclidean plane $\mathbb{R}^2$, we are asked to find a Steiner tree $T(l)$ such that at least one Steiner point is located at this line l, the objective is to minimize total weight of such a Euclidean Steiner tree $T(l)$, *i.e.*, $\min\{\sum_{e \in T(l)} w(e) | T(l)$ is an Euclidean Steiner tree as mentioned-above$\}$, where we define weight $w(e) = 0$ if the end-points u, v of each edge $e = uv \in T(l)$ are both located at this line l and otherwise we denote weight $w(e)$ to be the Euclidean distance between u and v.

We refer this Holby's problem as the 1-line-fixed Euclidean minimum Steiner tree (1LF-EMST) problem.

In particular, when Steiner points added are all located at this line l, we refer this problem as the constrained Euclidean minimum Steiner tree (CEMST) problem.

In addition, when there is exactly one Steiner point in this line l, such a constrained Euclidean minimum Steiner tree becomes a constrained Euclidean minimum spanning tree as shown in (Chen and Zhang [3]).

Holby [8] considered the second variation of the Euclidean minimum Steiner tree problem by finding the addition of a "Steiner line" in addition to Steiner points whose weight is not counted in the resulting network.

Given a set $P = \{r_1, r_2, \ldots, r_n\}$ of n points in $\mathbb{R}^2$, we are asked to find the location of a line l and a Steiner tree $T(l)$ such that at least one Steiner point is located at such a line l, the objective is to minimize total weight of such a Steiner tree $T(l)$, *i.e.*, $\min\{\sum_{e\in T(l)} w(e) | l$ is a line and $T(l)$ is an Euclidean Steiner tree as mentioned-above$\}$, where we define the weights of edges in such a Steiner tree $T(l)$ to be the same as mentioned-above.

We refer the second Holby's problem as the 1-line Euclidean minimum Steiner tree (1L-EMST) problem.

The 1L-EMST problem does not take any line l as an input, however, we shall find such a line l to minimize total weight $w(T(l))$.

The 1LF-EMST problem and the 1L-EMST problem have many applications in our reality life, such as transportation, communication, or location of some facilities.

As what Holby presented [8], given a highway l and n towns around this highway, we are asked to connect these n towns to such a highway l in order that the total weight of lengths is minimized.

In addition, given n towns at the plane, we are asked to find the location of a highway l (where such a highway will be built at the expense paid for by the federal government) such that these n towns are connected to such a highway l with the minimum total weight of lengths in a connected network.

These two problems can be modelled as the 1LF-EMST problem and the 1L-EMST problem mentioned-above, respectively.

3 Key Lemmas

Lemma 1 (Kruskal [10]) The minimum spanning tree problem can be optimally solved by the Kruskal algorithm in time $O(m \log n)$, where $n = |V(G)|$ and $m = |E(G)|$.

Using some strategy to construct a Delaunay triangulation (Berg et al. [1]) and introducing a single geometric structure called as the Voronoi diagram, Shamos and Hoey [14] designed an efficient exact algorithm (called as the Shamos-Hoey algorithm) to solve the Euclidean minimum spanning tree problem as follows.

Lemma 2 (Shamos and Hoey [14]) An Euclidean minimum spanning tree on n points in $\mathbb{R}^2$ can be constructed by an exact algorithm in time $O(n \log n)$.

It is natural to use an Euclidean minimum spanning tree to approximate an Euclidean minimum Steiner tree. The Euclidean Steiner ratio is the minimum upper-bound for the ratio between total lengths of an Euclidean minimum spanning tree and an Euclidean minimum Steiner tree for the same set P of n points in $\mathbb{R}^2$.

For this Euclidean Steiner ratio, Gilbert and Pollak [9] conjectured that the Euclidean Steiner ratio is $2/\sqrt{3} \approx 1.155$. By now, the best result for this Euclidean Steiner ratio is a result $(1/0.82416874.. \approx 1.214)$ due to Chung and Graham [4] shown as Lemma 3.

Lemma 3 (Chung and Graham [4]) Given a set P of n points in $\mathbb{R}^2$, denote by $L_M(P)$ the total lengths of an Euclidean minimum spanning tree and by $L_S(P)$ the total lengths of an Euclidean minimum Steiner tree on this set P, respectively. We have the fact $L_M(P) \leq 1.214 \cdot L_S(P)$.

Lemma 4

(1) Given a fixed line l and a set P of n points in $\mathbb{R}^2$ for the CEMST problem, if $\overline{r_i s_i}$ is one segment in an optimal constrained Euclidean minimum Steiner tree T, where r_i is a point in P and s_i is a point of T at the fixed line l, respectively, then this point s_i is exactly the vertical foot l_{r_i} from the point r_i to the line l.

(2) Given a fixed line l and a set P of n points in $\mathbb{R}^2$ for the 1LF-EMST problem, if $\overline{r_i s_i}$ is one segment in an optimal 1-line-fixed Euclidean minimum Steiner tree T, where r_i is either a point in P or a Steiner point of T that is not at the fixed line l and s_i is a point of T at the fixed line l, respectively, then this point s_i is exactly the vertical foot l_{r_i} from the point r_i to the line l.

4 The 1LF-EMST Problem

Using Lemma 4, we construct a constrained Euclidean minimum Steiner tree in the following strategies.

(1) Use the Shamos-Hoey algorithm [14] to produce an Euclidean minimum spanning tree $T = (P, E_T)$ on $P = \{r_1, r_2, \ldots, r_n\}$;

(2) Construct a weighted graph $G = (P \cup \{r_0\}, E, w)$, where r_0 is a new vertex to represent the fixed line l and $E = \{r_i r_j | \overline{r_i r_j} \in E_T$, where both points r_i and r_j are located at same side of the fixed line $l\} \cup \{r_i r_0 | r_i \in P\}$, and we denote $w(r_i r_j) = dist(r_i, r_j)$ for each $\overline{r_i r_j} \in E_T$ and $w(r_i r_0) = dist(r_i, l)$ $(= dist(r_i, l_{r_i}))$ for each $r_i \in P$;

(3) Use the Kruskal algorithm [10] to find a minimum spanning tree T_G in the weighted graph G, then construct a constrained Euclidean Steiner tree using the minimum spanning tree T_G.

Our algorithm $\mathcal{A}_{CEMST}$ to solve the CEMST problem is described as follows.

Algorithm $\mathcal{A}_{CEMST}$

Input: A fixed line l and a set $P = \{r_1, r_2, \ldots, r_n\}$ of n points in $\mathbb{R}^2$;

Output: a constrained Euclidean minimum Steiner tree $T(l)$.

Begin

Step 1 Denote $T(l) = (V^*, E^*)$, where $V^* = P$ and $E^* = \emptyset$;

Step 2 Use the Shamos-Hoey algorithm [14] to find an Euclidean minimum spanning tree $T = (P, E_T)$ on P;

Step 3 Construct a weighted graph $G = (P \cup \{r_0\}, E, w)$ as mentioned-above;

Step 4 Use the Kruskal algorithm [10] to find a minimum spanning tree T_G in G, corresponding to the weight function $w(\cdot)$;

Step 5 For each edge $e \in E(T_G)$ do

(1) If $(e = r_i r_j \in E(T_G)$, where $i \geq 1$ and $j \geq 1)$ then

we denote $E^* = E^* \cup \{\overline{r_i r_j}\}$

(2) If $(e = r_i r_0 \in E(T_G)$, where $i \geq 1)$ then

we denote $E^* = E^* \cup \{\overline{r_i l_{r_i}}\}$ and $V^* = V^* \cup \{l_{r_i}\}$;

Step 6 Output $T(l) = (V^*, E^*)$.

End

Using the algorithm $\mathcal{A}_{CEMST}$, we obtain the following

Theorem 1 The algorithm $\mathcal{A}_{CEMST}$ optimally solves the constrained Euclidean minimum Steiner tree (CEMST) problem in time $O(n\log n)$, where an instance of the CEMST problem consists of a fixed line l and a set $P=\{r_1,r_2,\ldots,r_n\}$ of n points in $\mathbb{R}^2$.

Using Lemma 3 due to Chung and Graham [4] for many times, we obtain the following

Theorem 2 The algorithm $\mathcal{A}_{CEMST}$ is also a 1.214-approximation algorithm in time $O(n \log n)$ to solve the 1LF-EMST problem, *i.e.*, given any instance P of n points in $\mathbb{R}^2$, the algorithm $\mathcal{A}_{CEMST}$ produces a 1-line-fixed Euclidean Steiner tree $T(l)$ to satisfy $w(T(l)) \leq 1.214 \cdot w(T_S^*(l))$, where $T_S^*(l)$ is an optimal solution for the instance P of the 1LF-EMST problem.

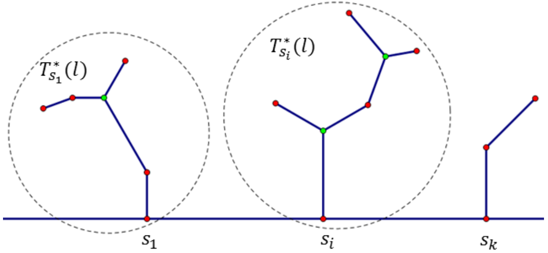

Figure 1: The relationship between $T(l)$ and $T^*_S(l)$

5 The 1L-EMST Problem

Definition 1 (Morris and Norback [11]) Given m "demand" points $Q=\{q_1,q_2,\ldots,q_m\}$ in $\mathbb{R}^2$ with coordinates $q_i=(x_i,y_i)$, the linear facility can be described by a line whose equation is $y=k_qx+b_q$, where (k_q,b_q) is an optimal solution to solve the following optimization problem

$$\min_{q\in Q} f_q(k,b)=\sum_{i=1}^{m}\frac{|y_i-kx_i-b|}{\sqrt{k^2+1}} \tag{1}$$

i.e., Formula 1 is equivalent to find a line in $\mathbb{R}^2$, whose equation is $y=k_qx+b_q$, such that the sum of distances from these m points q_1, q_2, …, q_m to this line $y=k_qx+b_q$ is minimized.

For an optimal solution of the optimization problem in Formula 1, we need the following lemma due to Morris and Norback [11], which can be found in polynomial time using an algorithm referred as the Morris-Norback algorithm.

Lemma 6 (Morris and Norback [11]) An optimal solution to the optimization problem as mentioned-above (seeing Formula 1) exists which is a line satisfying equation $y = k_q x + b_q$ to pass through at least two "demand" points in Q.

For the 1L-EMST problem, we have the following result that is similar to Lemma 6, which is an important result to find an optimal solution for the 1L-EMST problem.

Lemma 7 Given an instance of the 1L-EMST problem, there exists an optimum solution in which we can choose a line l to pass through at least two points in the set P.

Proof. We may assume that $T(l_1^*)$ is an optimal solution for the 1L-EMST problem, satisfying

(1) $w(T(l_1^*))$ is minimized among all 1-line Euclidean Steiner trees in $\mathbb{R}^2$, where a line l_1^* and a Steiner tree $T(l_1^*)$ are constructed,

(2) the line l_1^* contains points in the set P as many as possible, and

(3) $T(l_1^*)$ contains Steiner points being outside of the line l_1^* as less as possible.

Claim 1 This line l_1^* passes through at least one point in the set P.

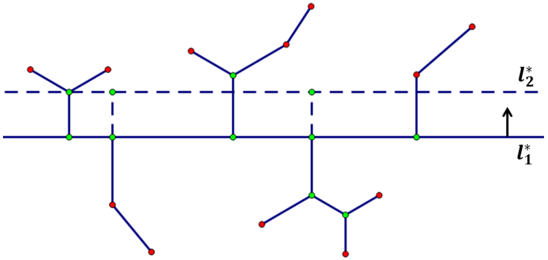

Figure 2: A process for first part in Claim 1

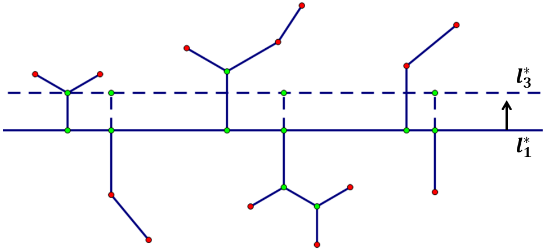

Figure 3: A process for second part in Claim 1

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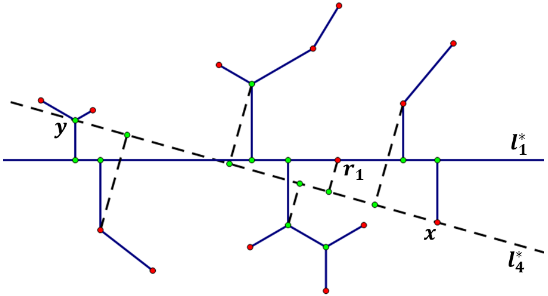

Figure 4: A process for a proof of Lemma 7

Basing on Lemma 7, we design an algorithm as follows, referred as the algorithm $\mathcal{A}_{1L-EMST}$, to solve the 1L-EMST problem.

Algorithm $\mathcal{A}_{1L-EMST}$

Input: A set $P = \{r_1, r_2, \ldots, r_n\}$ of n points in $\mathbb{R}^2$;

Output: a line l and a Steiner tree T.

Begin

Step 1 For $i = 1$ to n do:

For $j = 1$ to n $(j \neq i)$ do:

Choose a line $l_{i,j}$ to pass through r_i and r_j;

Use the algorithm $\mathcal{A}_{CEMST}$ on the set $P - \{r_i, r_j\}$ to construct a constrained Euclidean minimum Steiner tree $T(l_{i,j})$;

Step 2 Choose two points r_{i_0} and r_{j_0} to satisfy $w(T(l_{i_0,j_0}))$
$= \min\{w(T(l_{i,j}))|1 \le i \le n,\ 1 \le j \le n \text{ and } j \ne i\}$

Step 3 Output "the line l_{i_0,j_0} to pass through two points r_{i_0} and r_{j_0} and a 1-line Euclidean Steiner tree $T(l_{i_0,j_0})$."

End

Using the algorithm $\mathcal{A}_{1L-EMST}$, we obtain the following

Theorem 3 The algorithm $\mathcal{A}_{1L-EMST}$ is a 1.214-approximation algorithm for the 1L-EMST problem, and its time complexity is $O(n^3 \log n)$, where n is the number of points in $\mathbb{R}^2$.

6 Conclusion

In this talk, we consider the 1-line Euclidean minimum Steiner tree problem and its two special versions.

(1) Using a polynomial-time exact algorithm to find a constrained Euclidean minimum Steiner tree, we design a 1.214-approximation algorithm to solve the 1LF-EMST problem,

(2) Using a combination of a technique of finding linear facility location and an important Lemma 7, we can provide a 1.214-approximation algorithm to solve the 1L-EMST problem.

A challenging task for further research is to design another approximation algorithm to solve the 1L-EMST problem in lower time complexity. And it is another challenging to design some polynomial-time approximation schemes (PTASs) for these two problems.

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Thank You !