Rainbow Ramsey Number for Posets

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joint work with many people

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Introduction

Partially Ordered sets

A **poset (partially ordered set)** $P = (P, \leq)$ is a set P with a binary partial order relation $\leq$ satisfying

1. For all $x \in P$, $x \leq x$. (reflexivity)
2. If $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetry)
3. If $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

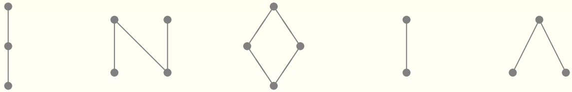


Figure: The Hasse diagrams of some small posets.

The **Boolean lattice** $\mathcal{B}_n$ is the poset whose elements are subsets of $[n]$ and the partial order is the inclusion relation on sets .

Partially Ordered sets

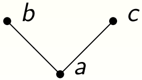
A poset $P_1 = (P_1, \leq_1)$ contains another poset $P_2 = (P_2, \leq_2)$ as an **(induced) subposet** if there is an injection $f : P_2 \to P_1$ such that

$$a \leq_2 b \Leftrightarrow f(a) \leq_1 f(b).$$

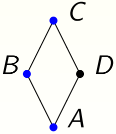
A poset $P_1 = (P_1, \leq_1)$ contains another poset $P_2 = (P_2, \leq_2)$ as a **(weak) subposet** if there is an injection $f : P_2 \to P_1$ such that

$$a \leq_2 b \Rightarrow f(a) \leq_1 f(b).$$

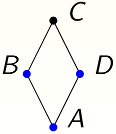
Example:



$$c \overset{f}{\longmapsto} C \qquad b \longmapsto B \qquad a \longmapsto A$$



$$c \overset{f'}{\longmapsto} D \qquad b \longmapsto B \qquad a \longmapsto A$$



Poset Ramsey Number

Definition

Given posets P and Q, the **strong Ramsey number** $R^*(P, Q)$ *is the minimum n such that any 2-coloring (red/blue) on $\mathcal{B}_n$ contains either a red P or a blue Q as an induced subposet.*

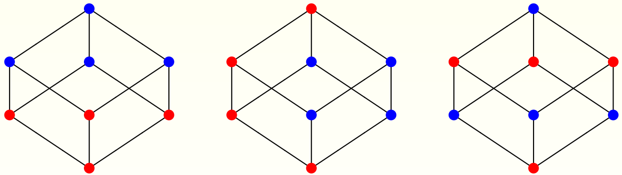


Figure: Three colorings on $\mathcal{B}_3$ without a monochromatic Q_2.

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Poset Ramsey Number

Theorem (Axenovich and Walzer, 2017)

For hypercubes (Boolean posets) Q_n, Q_m,

$$\text{(i) } 2n \leq R^*(Q_n, Q_n) \leq n^2 + 2n,$$
$$\text{(ii) } R^*(Q_2, Q_2) = 4,\ R^*(Q_3, Q_3) \in \{7, 8\},$$
$$\text{(iii) } R^*(Q_1, Q_n) = n + 1,\ R^*(Q_2, Q_n) \leq 2n + 2,$$
$$\text{(iv) } R^*(Q_n, Q_m) \leq mn + n + m.$$

Remark. The strong Ramsey number $R^*(P_1, \ldots, P_k)$ for posets $P_1, \ldots, P_k$ can be defined analogously. If $P_1 = \cdots = P_k = P$, then we use $R_k^*(P)$ to denote $R^*(P_1, \ldots, P_k)$.

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Poset Ramsey Number

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$$(iv)\ R^*(Q_n, Q_m) \le mn + n + m.$$

Theorem (Axenovich and Walzer, 2017)

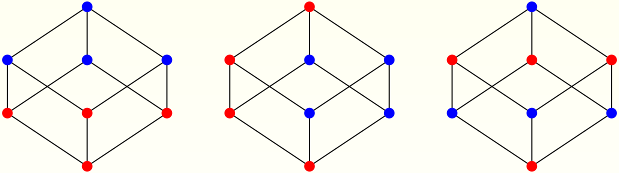
For any poset P,

$$R_k^*(P) = \Theta(k).$$

Poset Ramsey Number

Definition

Given posets P and Q, the **weak Ramsey number** $R(P, Q)$ *is the minimum n such that any 2-coloring (red/blue) on $\mathcal{B}_n$ contains either a red P or a blue Q as a weak subposet.*



Remark. To see more results of the weak version of Ramsey number for posets, please see "Ramsey number for partially-ordered posets" by Cox and Stolee in *Order* 35(3), pp 557–579.

The Results

The Results

Theorem (Wu, 2018)

For the posets P with $|P| = 4$, we have the following results:

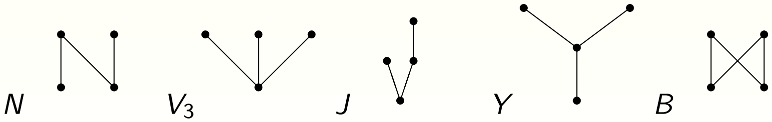
(i) $R^*(N, N) = 4$,

(ii) $R^*(V_3, V_3) = 5$,

(iii) $R^*(J, J) = 5$,

(iv) $R^*(Y, Y) = 5$, *and*

(v) $R^*(B, B) = 6$.



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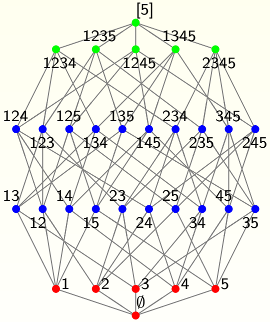
The Results

Theorem (Chen, Cheng, L. and Liu, 2018+)

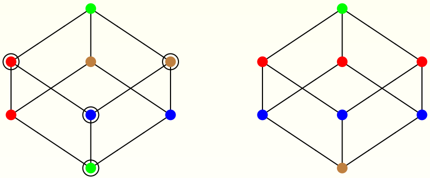
For the poset Q_2, we have $R^*_3(Q_2) = 6$.



The Results

Definition

Let P be a poset, and let f be a coloring on the Boolean lattice $\mathcal{B}_n$. If $\mathcal{B}_n$ contains P as an induced subposet with different colors on different elements, then we say $\mathcal{B}_n$ contains a rainbow P under f.



The $\mathcal{B}_3$ on the left contains a rainbow Y under the coloring.

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Definition

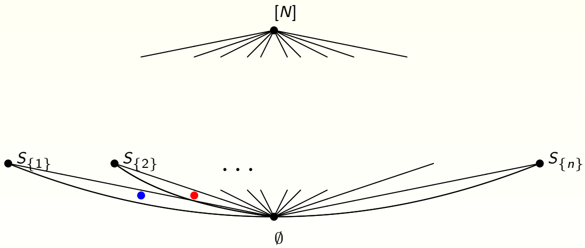
Given two posets P and Q, the **strong rainbow Ramsey number** *for posets P and Q, $RR^*(P, Q)$, is the minimum number n such that for any coloring f on $\mathcal{B}_n$, either there is a monochromatic P or a rainbow Q as an induced subposet.*

The Results

Theorem (Chen, Cheng, L., Liu, 2018+)

$n(2^m-1) \le RR^*(Q_n, Q_m) \le (2^m-1)R^*_{2^m-1}(Q_n).$

Proof. Let $N=(2^m-1)R_{2^m-1}(Q_n)$. For any coloring f on $\mathcal{B}_N$, we assume there is no monochromatic Q_n in $\mathcal{B}_N$, and show that there is a rainbow Q_m under f. Write $[N]=\bigcup_{I:\emptyset\neq I\subseteq[m]} S_I$ with $S_I=|R_{2^m-1}(Q_n)|$.



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Theorem (Chen, Cheng, L., Liu, 2018+)

$n(2^m - 1) \leq RR^*(Q_n, Q_m) \leq (2^m - 1)R^*_{2^m-1}(Q_n).$

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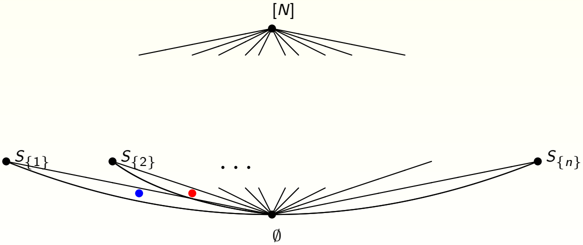
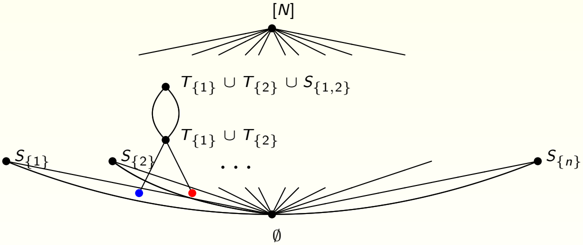
For $|I| = 1$, since if $f|_{2^{[S_I]}}$ does not contain a monochromatic Q_n, then there are at least 2^m colors on the subsets in $2^{[S_I]}$. Then for these I's, we pick a nonempty set $T_I \subseteq S_I$ so that $f(T_I)$'s are all distinct.



The Results

For sets in the interval $[T_{\{1\}} \cup T_{\{2\}}, T_{\{1\}} \cup T_{\{2\}} \cup S_{\{1,2\}}]$, there are at least 2^m colors on the sets in the interval, since B_N does not contain a monochromatic Q_n. So we can pick one set whose color is different from $f(T_I)$'s and $f(\emptyset)$ as denote it as $T_{\{1,2\}}$.



Repeat this method from the small subsets to large subsets, we can construct a rainbow Q_m. □

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The Results

Exact Values of $RR^*(Q_n, Q_m)$ for some n and m.

THEOREM (Chen, Cheng, L., Liu, 2018+)

$RR^*(Q_n, Q_1) = n.$

THEOREM (Chen, Cheng, L., Liu, 2018+)

$RR^*(Q_1, Q_n) = 2^n - 1.$

THEOREM (Chen, Cheng, L., Liu, 2018+)

$RR^*(Q_2, Q_2) = 6.$

The Results

Given a family $\mathcal{F}$ of subsets of $[n]$, the **Lubell function** of $\mathcal{F}$ is defined to be

$$\bar{h}_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}.$$

Let $e(P)$ be the maximum number such that the union of any $e(P)$ consecutive levels in any Boolean lattice does not contain P as a weak subposet.

Definition

A poset P is **uniformly Lubell bounded** *if for any n, every family $\mathcal{F}$ of subsets of $[n]$, which does not contain P as a weak subposet satisfies $\bar{h}_n(\mathcal{F}) \leq e(P)$.*

The Results

Theorem (CGLMNPV, 2018+)

Let P be a uniformly Lubell bounded poset and $\mathcal{F}$ be a family of subsets with $\bar{h}_n(\mathcal{F}) > e(P)(k-1)$. Then any coloring c on $\mathcal{F}$ contains either a monochromatic P or a raibow chain P_k as a weak subposet.

Becasue $\bar{h}_n(\mathcal{B}_n) = \sum_{F\subseteq[n]} \frac{1}{\binom{n}{|F|}} = n+1$ and P_k contains any k-element poset as a weak subpost , the theorem implies the following corollary immediately.

Corollary (CGLMNPV, 2018+)

If P is uniformly Lubell-bounded, then $RR(P,Q) = e(P)(|Q|-1)$ holds for any poset Q.

The Results

Proof of the theorem. We prove by induction on k.
For $k = 1$, if we color a nonempty family of subsets of $[n]$, then at least one color class is not empty. So there is a monochromatic P_1 (singleton).
Suppose this holds for some integer k. Let us color a family $\mathcal{F}$ with $\bar{h}_n(\mathcal{F}) > e(P)k$, and then apply the "**min-max partition**" on the set of full chains in $\mathcal{B}_n$ to get a subfamly $\mathcal{F}'$ with $\bar{h}_m(\mathcal{F}') > e(P)k$.

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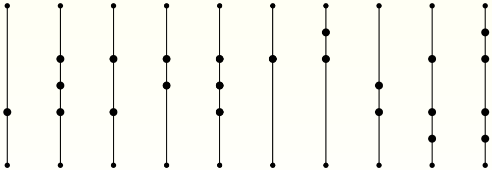

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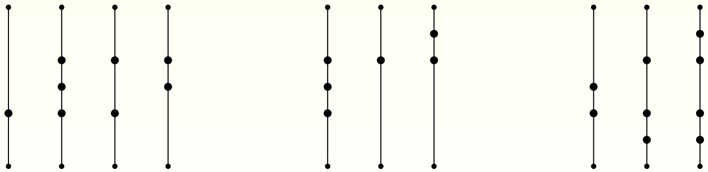
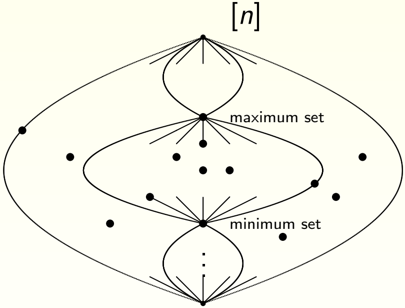


The Results

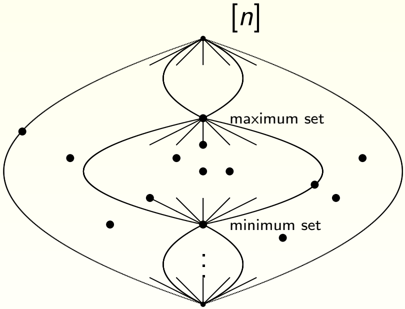
This family contains a minimum and a maximum subset. Let the minimum subset be colored by 1, and let $\mathcal{F}_1$ be the subfamily of $\mathcal{F}$, which contains all subsets of color 1.



If $\mathcal{F}_1$ does not contain P as a weak subposet, then $\bar{h}_m(\mathcal{F}_1) < e(P)$ and $\bar{h}_m(\mathcal{F}' - \mathcal{F}_1) > e(P)(k-1)$.

The Results

By induction, either $\mathcal{F}' - \mathcal{F}_1$ contains a monochromatic P as a weak subposet, or it contains a rainbow P_{k-1} as a weak subposet. Assume that the latter case happens.



Then the rainbow P_{k-1} does not contain subsets of color 1. We elongate the rainbow P_{k-1} by adding the minimum subset. □

Future work

- $R_k^*(Q_2) = 2k$?
- Good estimations of $R^*(Q_n, Q_m)$.
- Other type of Ramsey Problems on Posets.

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Thank you for your attention.