Some recent results on enumeration of tableaux and lattice paths

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Outline

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Integer partitions

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partititon of n, i.e.

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n,$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$.

The Ferrers diagram of λ is a left-justfied array of cells with λ_i cells in the i-th row, for $1 \leq i \leq k$.

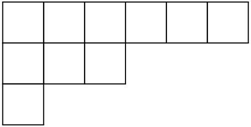


Figure: The Ferrers diagram of a partition $\lambda = (6, 3, 1) \vdash 10$.

Semistandard Young tableau and standard Young tableau

A semistandard Young tableau (SSYT) of shape λ is a filling of the Ferrers diagram of λ with positive integers such that every row is strictly increasing and every column is weakly increasing.

A standard Young tableau (SYT) of shape $\lambda \vdash n$ is a filling of the Ferrers diagram of λ with $\{1, 2, \ldots, n\}$ such that every row and column is strictly increasing.

2	4	6	7	8	9
4	5	6			
8					

1	3	4	5	8	10
2	6	7			
9					

Figure: A semi-standard Young tableau of shape $(6, 3, 1)$ and a standard Young tableau of shape $(6, 3, 1)$.

Major index and amajor index of a tablau

A descent of an SSYT T is any instance of i followed by an $i+1$ in a lower row of T. $D(T)$: the descent set of T. The major index of T is defined by $\mathrm{maj}(T) = \sum_{i \in D(T)} i$. An ascent of T is any instance of i followed by an $i+1$ in a higher row of T than i. $A(T)$: the ascent set of T. The amajor index of T is defined by $\mathrm{amaj}(T) = \sum_{i \in A(T)} i$.

1	**2**	**5**	10
3	4	**8**	
6			
7			
9			

1	2	5	10
3	**4**	8	
6			
7			
9			

Figure: $T \in \mathrm{SYT}(4,3,1,1,1)$.

$D(T) = \{2,5,6,8\}$, $\mathrm{maj}(T) = 21$. $A(T) = \{4,7,9\}$, $\mathrm{amaj}(T) = 20$.

Major index for standard Young tableaux

Lemma (Stanley's q-hook length formula)
For any partition $\lambda = \sum_i \lambda_i$ of n, we have

$$\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \frac{q^{b(\lambda)}[n]!}{\prod_{u \in \lambda} h(u)}. \tag{1}$$

Here $b(\lambda) = \sum_i (i-1)\lambda_i$.

The famous RSK algorithm is a bijection between permutations of length n and pairs of SYTs of order n of the same shape. Under this bijection, the descent set of a permutation is transferred to the descent set of the corresponding "recording tableau". Therefore many problems involving the statistics descent or major index of pattern-avoiding permutations can be translated to the study of descent or major index of tableaux.

Standard Young tableaux of shape $2 \times n$

For any positive integer n, we have

$$C_q(n) = \sum_{T \in \mathrm{SYT}(2\times n)} q^{\mathrm{maj}(T)} = \frac{q^n}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}. \qquad (2)$$

Here $[n] = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$, $[n]! = [n][n-1]\cdots[1]$ and $\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}$.

For example, when $n = 3$, we have

$$C_q(3) = \frac{q^3}{[3+1]} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = q^3 + q^5 + q^6 + q^7 + q^9.$$

And there are five SYT of shape 2×3, with major index 3,6,7,5,9.

1	2	**3**
4	5	6

1	**2**	**4**
3	5	6

1	**2**	**5**
3	4	6

1	3	**4**
2	5	6

1	**3**	**5**
2	4	6

Increasing tableaux

An increasing tableau is an SSYT such that both rows and columns are strictly increasing, and the set of entries is an initial segment of positive integers (if an integer i appears, positive integers less than i all appear).

We denote by $\mathrm{Inc}_k(\lambda)$ the set of increasing tableaux of shape λ with entries are $\{1, 2, \ldots, n-k\}$.

1	**2**	3
2	4	5

1	**2**	4
2	3	5

1	2	**3**
3	4	5

1	2	**4**
3	**4**	5

1	3	**4**
2	**4**	5

Figure: There are five increasing tableaux in $\mathrm{Inc}_1(2 \times 3)$.

Increasing tableau is defined by O. Pechenik who studied increasing tableaux in $\mathrm{Inc}_k(2 \times n)$, i.e., increasing tableaux of shape $2 \times n$, with exactly k numbers appeared twice.

O. Pechenik, Cyclic Sieving of Increasing Tableaux and Small Schröder Paths. *J. Combin. Theory Ser. A*, 125: 357–378, 2014.

Major index for Increasing tableau of shape $2 \times n$

Theorem (O. Pechenik)
For any positive integer n, and $0 \le k \le n$ we have

$$S_q(n,k) = \sum_{T \in \mathrm{Inc}_k(2\times n)} q^{\mathrm{maj}(T)} = \frac{q^{n+k(k+1)/2}}{[n+1]} \begin{bmatrix} n-1 \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix}. \qquad (3)$$

For example, when $n = 3,\ k = 1$ we have

$$\begin{array}{|c|c|c|}\hline \mathbf{1} & 2 & \mathbf{3} \\ \hline 2 & 4 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \hline 2 & 3 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & \mathbf{2} & \mathbf{3} \\ \hline 3 & 4 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & \mathbf{2} & \mathbf{4} \\ \hline 3 & 4 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline \mathbf{1} & \mathbf{3} & \mathbf{4} \\ \hline 2 & 4 & 5 \\ \hline \end{array}$$

$$S_q(3,1) = \sum_{T \in \mathrm{Inc}_1(2\times 3)} q^{\mathrm{maj}(T)} = \frac{q^4}{[3+1]} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = q^8 + q^7 + q^6 + q^5 + q^4.$$

Pechinik's proof involves the cyclic sieving method.

A refinement of small Schröder number

Setting $q = 1$, we get the cardinality of $\mathrm{Inc}_k(2 \times n)$:

$$s(n,k) = \frac{1}{n+1}\binom{n-1}{k}\binom{2n-k}{n}. \tag{4}$$

$s(n,k)$ is considered as a refinement of the small Schröder number which counts the following sets:

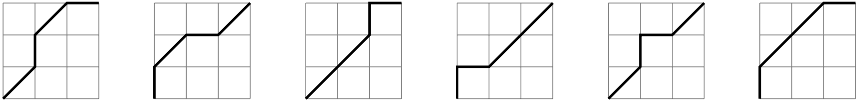
1. Dissections of a convex $(n+2)$-gon into $n-k$ regions;
2. SYTs of shape $(n-k, n-k, 1^k)$;
3. small Schröder n-paths with k flat steps.

In 1996 Stanley gave a bijection between the first two sets.

R. P. Stanley, Polygon dissections and standard Young tableaux. *J. Combin. Theory Ser. A*, 76: 175–177, 1996.

Schröder paths

A Schröder n-path is a lattice path goes from $(0,0)$ to (n,n) with steps $(0,1)$, $(1,0)$ and $(1,1)$ and never goes below the diagonal line $y=x$. If there is no F steps on the diagonal line, it is called a small Schröder path.



There is an obvious bijection between Schröder n-paths and SSYTs of shape $2\times n$: read the numbers i from 1 to $2n-k$ in increasing order, if i appears only in row 1 (2), it corresponds to a U (D) step, if i appears in both rows, it corresponds to an F step.

1	2	**3**
1	**3**	4

1	**2**	**4**
2	3	**4**

1	**2**	3
1	**2**	4

1	**3**	**4**
2	**3**	**4**

1	2	**4**
1	3	**4**

1	**2**	**3**
2	**3**	4

Motivation: are there any interesting result for these tableaux that correspond to all Schröder n-paths?

Row-increasing tableaux

A row-increasing tableau is an SSYT with strictly increasing rows and weakly increasing columns, and the set of entries is a consecutive segment of positive integers.

We denote by $\mathrm{RInc}_k^m(\lambda)$ the set of row-increasing tableaux of shape λ with set of entries $\{m+1, m+2, \ldots, m+n-k\}$. When $m=0$, we will just denote $\mathrm{RInc}_k^0(\lambda)$ as $\mathrm{RInc}_k(\lambda)$. It is obvious that $\mathrm{Inc}_k(\lambda) \subseteq \mathrm{RInc}_k(\lambda)$.

$$
\begin{array}{|c|c|c|}\hline \mathbf{1} & 2 & \mathbf{3} \\ \hline \mathbf{1} & \mathbf{3} & 4 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline 1 & \mathbf{2} & \mathbf{4} \\ \hline \mathbf{2} & 3 & \mathbf{4} \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline \mathbf{1} & \mathbf{2} & 3 \\ \hline \mathbf{1} & \mathbf{2} & 4 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline 1 & \mathbf{3} & \mathbf{4} \\ \hline 2 & \mathbf{3} & \mathbf{4} \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline \mathbf{1} & 2 & \mathbf{4} \\ \hline \mathbf{1} & 3 & \mathbf{4} \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline 1 & \mathbf{2} & \mathbf{3} \\ \hline \mathbf{2} & \mathbf{3} & 4 \\ \hline \end{array}
$$

Figure: There are 6 row-increasing tableaux in $\mathrm{RInc}_2(2 \times 3)$.

It is not hard to show that $\mathrm{RInc}_k(2 \times n)$ is counted by

$$r(n,k) = \frac{1}{n-k+1}\binom{2n-k}{k}\binom{2n-2k}{n-k}. \tag{5}$$

$r(n,k)$ is considered as a refinement of the large Schröder number.

Major index polynomial for $\mathrm{RInc}_k(2 \times n)$

There is a bijection $f\colon \mathrm{RInc}_k(2 \times n) \backslash \mathrm{Inc}_k(2 \times n) \mapsto \mathrm{Inc}_{k-1}(2 \times n)$.

Given $T \in \mathrm{RInc}_k(2 \times n) \backslash \mathrm{Inc}_k(2 \times n)$, find the minimal integer j such that $T_{1,j} = T_{2,j}$,. Now we first delete the entry $T_{2,j}$, then move all the entries on the right of $T_{2,j}$ one box to the left and set the last entry as $2n-k+1$, and define the resulting tableau to be $f(T)$.

$$T: \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 3 & 4 & 6 & 7 \\ \hline \end{array} \quad \mapsto \quad f(T): \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 4 & 6 & 7 & 8 \\ \hline \end{array}$$

Figure: An example of f with $T \in \mathrm{RInc}_3(2 \times 5) \backslash \mathrm{Inc}_3(2 \times 5)$ and $f(T) \in \mathrm{Inc}_2(2 \times 5)$.

However, f does NOT preserve the major index. In fact we have

Theorem

For any positive integer n, k with $k < n$, we have

$$R_q(n,k) = S_q(n,k) + S_q(n,k-1) + (1-q^{2n-k})(S_q(n-1,k-1) + S_q(n-1,k-2)).$$

Major index polynomial for $\mathrm{RInc}_k(2\times n)$

Theorem (O. Pechenik)

There exists a bijection γ between $\mathrm{Inc}_k(2\times n)$ and $\mathrm{SYT}(n-k, n-k, 1^k)$ which preserves the descent set.

E.g., we have $A=\{4,6,8\}$ and $B=\{6,7,9\}$.

1	2	4	5	6	8
3	4	6	7	8	9

1	2	5
3	4	8
6		
7		
9		

Theorem

For any positive integer n, and $0\le k\le n$ we have

$$R_q(n,k)=\sum_{T\in\mathrm{RInc}_k(2\times n)} q^{\mathrm{maj}(T)}=\frac{q^{n+k(k-3)/2}}{[n-k+1]}\begin{bmatrix}2n-k\\ k\end{bmatrix}\begin{bmatrix}2n-2k\\ n-k\end{bmatrix}. \quad (6)$$

Row-increasing tableaux of shape $(n-a,a)$

Major index polynomial for row-increasing tableaux of shape $(n-a,a)$:

$$R_{(n-a,a),k}(q) = q^{a+k(k-3)/2}\frac{[n-2a+1]}{[n-a-k+1]}\begin{bmatrix} n-k \\ k \end{bmatrix}\begin{bmatrix} n-2k \\ a-k \end{bmatrix}.$$

Summing over a, we get the major index polynomial for all row-increasing tableaux with n cells and at most two rows:

$$\sum_{a=k}^{\lfloor \frac{n}{2} \rfloor} R_{(n-a,a),k}(q) = q^{k(k-1)/2}\begin{bmatrix} n-k \\ k \end{bmatrix}\begin{bmatrix} n-2k \\ \lfloor \frac{n}{2} \rfloor - k \end{bmatrix}.$$

The result for increasing tableaux is more complicated. For example, we have

$$S_{(n-a,a),k} = \frac{2a^2-3na-a+n^2+n-k}{(n-a+1)(n-a)}\binom{n-k}{k}\binom{n-2k}{a-k}.$$

and

$$\sum_{a=k}^{\lfloor \frac{n}{2} \rfloor} S_{(n-a,a),k} = \frac{\lceil \frac{n}{2} \rceil - k}{\lceil \frac{n}{2} \rceil}\binom{n-k}{k}\binom{n-2k}{\lfloor \frac{n}{2} \rfloor - k}.$$

Amajor index polynomial for $\mathrm{RInc}_k(2 \times n)$

We also study the amajor index polynomial of SSYTs in $\mathrm{RInc}_k(2 \times n)$.

$$\widetilde{R}_q(n,k) = \sum_{T \in \mathrm{RInc}_k(2\times n)} q^{\mathrm{amaj}(T)} = \frac{q^{k(k-1)/2}}{[n-k+1]} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \begin{bmatrix} 2n-2k \\ n-k \end{bmatrix}.$$

We will prove the above formula by showing that

$$\sum_{T \in \mathrm{RInc}_k(2\times n)} q^{\mathrm{maj}(T)} = q^{n-k} \cdot \sum_{T \in \mathrm{RInc}_k(2\times n)} q^{\mathrm{amaj}(T)}.$$

For example, there are 6 row-increasing tableaux in $\mathrm{RInc}_2(2 \times 3)$, with the $(\mathrm{maj}, \mathrm{amaj})$ pairs $(4,3), (4,5), (2,4), (5,1), (3,3), (6,2)$.

1	2	3
1	2	4

1	3	4
2	3	4

1	2	4
1	3	4

1	2	3
1	3	4

1	2	4
2	3	4

1	2	3
2	3	4

We want to establish a bijection $\Phi : \mathrm{RInc}_k(2 \times n) \mapsto \mathrm{RInc}_k(2 \times n)$ such that

$$\mathrm{maj}(\Phi(T)) = \mathrm{amaj}(T) + n - k.$$

The general result

Theorem

There is a bijection $\Phi : \mathrm{RInc}_k(2 \times n) \mapsto \mathrm{RInc}_k(2 \times n)$ that preserves the second row, and

$$\mathrm{maj}(\Phi(T)) = \mathrm{amaj}(T) + n - k.$$

T :

1	**2**	4	5	**6**	**9**	10	12	**13**	14	**16**	18	**20**
2	3	**6**	7	8	**9**	11	**13**	15	**16**	17	19	**20**

			1	4	5			10	12	14		18
2	3	6	7	8	9	11	13	15	16	17	19	20

$\Phi(T)$:

1	**3**	4	5	**8**	**9**	10	**11**	12	14	**15**	18	**19**
2	**3**	6	7	**8**	**9**	**11**	13	**15**	16	17	**19**	20

Figure: An example of the map Φ with $n = 13$, $k = 6$, and $l = 3$.

The prime case

A row-increasing tableau T is prime if for each integer j satisfies $T_{1,j+1} = T_{2,j} + 1$, $T_{2,j+1}$ also appears in row 1 in T.

$\mathrm{pRInc}_k^m(\lambda)$: prime row-increasing tableaux of shape λ with set of entries $\{m+1, m+2, \ldots, m+n-k\}$.

For each $T \in \mathrm{pRInc}_k^m(2 \times n)$, let A be the set of numbers that appear twice, and B be the set of numbers that appear in the second row immediately left of an element of A in cyclic order.

Let $g(T)$ be the tableau of shape $2 \times n$ obtained by first deleting all elements in A from the first row and then inserting all elements in B into the first row and list them in increasing order, and keep the entries in row 2 unchanged.

In the following example, we have $A = \{2, 6, 9\}$ and $B = \{3, 8, 9\}$.

$$T : \begin{array}{|c|c|c|c|c|c|} \hline 1 & \mathbf{2} & 4 & 5 & \mathbf{6} & \mathbf{9} \\ \hline \mathbf{2} & 3 & \mathbf{6} & 7 & 8 & \mathbf{9} \\ \hline \end{array} \quad \xrightarrow{g} \quad g(T) : \begin{array}{|c|c|c|c|c|c|} \hline 1 & \mathbf{3} & 4 & 5 & \mathbf{8} & \mathbf{9} \\ \hline 2 & \mathbf{3} & 6 & 7 & \mathbf{8} & \mathbf{9} \\ \hline \end{array}$$

Lemma

The map g is an injection from $\mathrm{pRInc}_k^m(2 \times n)$ to $\mathrm{RInc}_k^m(2 \times n)$ which satisfies the following:

1) If $T_{2,1}$ appears only once in T, then $g(T)_{1,i+1} \leq g(T)_{2,i}$ for each $i, 1 \leq i \leq n-1$;
2) $T_{2,1}$ appears twice in T if and only if $g(T)_{1,n} = g(T)_{2,n}$.

Sketch of Proof: there are two cases:

- $T_{2,1}$ appears only once in T;

T :

5	7	**8**	10	11	**12**
6	**8**	9	**12**	13	14

$\xrightarrow{g}$

$g(T)$:

5	**6**	**7**	**9**	10	11
6	8	**9**	12	13	14

Lemma

The map g is an injection from $\mathrm{pRInc}_k^m(2 \times n)$ to $\mathrm{RInc}_k^m(2 \times n)$ which satisfies the following:

1) If $T_{2,1}$ appears only once in T, then $g(T)_{1,i+1} \leq g(T)_{2,i}$ for each $i, 1 \leq i \leq n-1$;
2) $T_{2,1}$ appears twice in T if and only if $g(T)_{1,n} = g(T)_{2,n}$.

Sketch of Proof: there are two cases:

- $T_{2,1}$ appears twice in T;

$T:$

1	**2**	4	5	**6**	**9**
2	3	**6**	7	8	**9**

$\xrightarrow{g}$

$g(T):$

1	**3**	4	5	**8**	**9**
2	**3**	6	7	**8**	**9**

$\tilde{T}:$

1	**2**	**3**	4	5	**8**
2	**3**	6	7	**8**	9

Lemma

For each $T \in \mathrm{pRInc}_k^m(2 \times n)$ we have

$$\mathrm{maj}(g(T)) = \begin{cases} \mathrm{amaj}(T) + n - k, & \text{if } T_{1,1} = T_{2,1}; \\ \mathrm{amaj}(T) + m + n - k, & \text{if } T_{1,1} \neq T_{2,1}. \end{cases} \tag{7}$$

T :

5	6	**8**	9	10	**13**
7	**8**	11	12	**13**	14

$\xrightarrow{g}$

$g(T)$:

5	6	**7**	9	10	**12**
7	8	11	**12**	13	14

T^0 :

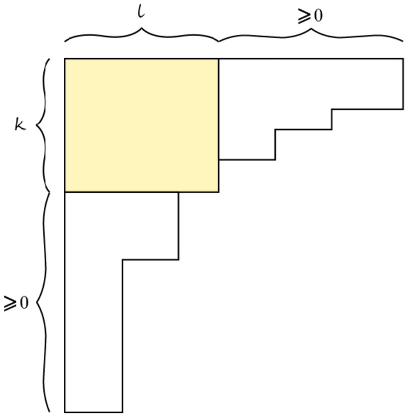
		5	6	9	10
7	8	11	12	13	14

1. $D(g(T)) \setminus D(T^0) = A(T) \setminus A(T^0)$.
   and therefore $\mathrm{maj}(g(T)) - \mathrm{maj}(T^0) = \mathrm{amaj}(T) - \mathrm{amaj}(T^0)$;
2. When $T_{1,1} \neq T_{2,1}$, $\mathrm{maj}(T^0) = \mathrm{amaj}(T^0) + m + n - k$.

Standard Young tableaux in a (k, l)-hook

f^λ : the number of SYTs of shape λ;

$\mathcal{H}(k, l; n) = \{\lambda = (\lambda_1, \lambda_2, \cdots)|\lambda \vdash n, \lambda_{k+1} \leq l\}$;



$S(k, l; n) = \sum_{\lambda \in \mathcal{H}(k,l;n)} f^\lambda$: number of SYTs in a (k, l)-hook.

Known results on $S(k,l;n)$ for small k and l

For the "stripe" case, it is known that

$$S(2,0;n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}; \quad S(3,0;n) = \sum_{j \geq 0} \frac{1}{j+1} \binom{n}{2j} \binom{2j}{j}.$$

$$S(4,0;n) = C_{\lfloor \frac{n+1}{2} \rfloor} \cdot C_{\lceil \frac{n+1}{2} \rceil}, \text{ and } \quad S(5,0;n) = 6 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} \cdot C_i \cdot \frac{(2i+2)!}{(i+2)!(i+3)!}.$$

Here $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

D. Gouyou-Beauchamps, Standard Young tableaux of height 4 and 5, *European J. Combin.* **10** 1989 69–82.

For the "hook" sums, only $S(1,1;n)$ and $S(2,1;n) = S(1,2;n)$ is known. It is easy to see that $S(1,1;n) = 2^{n-1}$. A. Regev proved that

$$S(2,1;n) = S(1,2;n) = \frac{1}{2} \sum_{j \geq 1} \binom{n}{j} \binom{n-j}{j} + 1, \tag{8}$$

and pointed out that this is related to Motzkin paths.

Counting SYTs and Motzkin paths

A Motzkin path of order n is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0,0)$ to $(n,0)$, using up-steps $(1,1)$, down-steps $(1,-1)$ and flat-steps $(1,0)$ that never goes below the x-axis.

It is a well-known result that the cardinality of $\mathcal{M}_n$ is the n-th Motzkin number. Note that $S(3,0;n)$ is exactly the n-th Motzkin number, and a nice bijection between $\mathcal{S}(3,0;n)$ and $\mathcal{M}_n$ is given by Eu.

S. Eu, Skew-standard Tableaux with Three Rows, *Adv. Appl. Math.* **45(4)** 2010, 463–469.

A hump in a Motzkin path is an up step followed by zero or more flat steps followed by a down step. If such a path is allowed to go below the x-axis, it is called a super Motzkin path. $\mathcal{SM}_n$: super Motzkin paths of order n, hm_n: total number of humps in all Motzkin paths of order n.

Motzkin paths, humps, and sumper Motzkin paths

A. Regev (2010) counted hm_n by a recurrence relation and the WZ method and found the following interesting relations and asked for combinatorial proofs.

$$hm_n = \frac{1}{2}(\#\mathcal{SM}_n - 1); \tag{9}$$

$$S(2,1;n) = hm_n + 1. \tag{10}$$

Ding and Du (2012) gave bijective proofs of this equations, and also proved that similar relation holds for Schröder Paths.

Later Mansour and Shattuck (2013) extended this study to (k,a)-paths and proved that similar relation holds between humps in (k,a)-paths and super (k,a)-paths by studying their generating functions.

$$(k+1)\sum_{P\in\mathcal{P}_n(k,a)} \#\mathrm{Humps}(P) = |\mathcal{SP}_n(k,a)| - \delta_{a|n}, \tag{11}$$

where $\delta_{a|n} = 1$ if a divides n or 0 otherwise.

Bijective proofs were given by Du, Nie and Sun (2015) and Yan (2015).

$S(2,1;n)$ and super Motzkin paths

Combining the above two equations

$$hm_n = \frac{1}{2}(\#\mathcal{SM}_n - 1);$$
$$S(2,1;n) = hm_n + 1.$$

we get

$$S(2,1;n) = \frac{1}{2}(\#\mathcal{SM}_n + 1). \tag{12}$$

$\mathcal{SM}_n^{*D}$: super Motzkin paths in $\mathcal{SM}_n$ whose last non-flat step (if any) is a down step. It is easy to see that

$$\#\mathcal{SM}_n^{*D} = \frac{1}{2}(\#\mathcal{SM}_n + 1) = \frac{1}{2}\sum_{j\geq 1}\binom{n}{j}\binom{n-j}{j} + 1. \tag{13}$$

Our main idea is to give a one-to-one correspondence between $\mathcal{S}(2,1;n)$ and $\mathcal{SM}_n^{*D}$.

Tight 012-words

Let $W = w_1 w_2 \cdots w_n$ be a word of length n on the alphabet $\{0, 1, 2\}$. For each integer $k = 0, 1, 2$, let $f_k(W)$ denote the number of k's in $w_1, w_2, \ldots, w_n$. We say that W is a tight 012-word of length n if

1) $f_0(w_1 w_2 \ldots w_j) \geq f_2(w_1 w_2 \ldots w_j)$, $j = 1, 2, \ldots, n$;
2) For each $j = 1, 2, \ldots, n-1$, $f_0(w_1 w_2 \ldots w_j) > f_2(w_1 w_2 \ldots w_j)$ when $w_{j+1} = 1$.

$\mathcal{W}_n$: all tight 012-words of length n. $\mathcal{W}_n^{*2}$: all tight 012-words of length n whose last nonzero number is 2.

For example, when $n = 4$, there are 19 tight 012-words:

0000, 0001, 0002, 0010, 0011, 0012, 0020, 0021, 0022, 0100, 0101, 0110, 0111, 0102, 0112, 0120, 0200, 0201, 0202,

Theorem

For any positive integer n, there is a bijection $\phi : \mathcal{S}(2, 1; n) \mapsto \mathcal{W}_n^{*2}$.

SYT and the corresponding tight 012-word

1	2	4	7	10
3	5	9	12	
6				
8				
11				

$$\mapsto \qquad W = 0\ 0\ 2\ 0\ 1\ 1\ 0\ 2\ 1\ 0\ 2\ 2$$

Figure: A standard Young tableaux and the corresponding tight 012-word.

$A = \{a_1, a_2, \ldots, a_s\}_<$: the set of numbers appear in row 1 of T;

$B = \{b_1, b_2, \ldots, b_{n-s}\}_< = [n] \setminus A$.

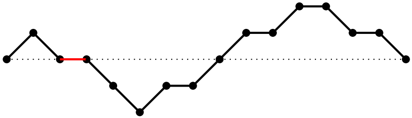
$\phi(T) = W = w_1 w_2 \cdots w_n$ such that:

$w_{a_1} = w_{a_2} = \cdots = w_{a_s} = 0$, $w_{b_{n-s}} = 2$, and for each $i = 1, 2, \ldots, n-s-1$, we set $w_{b_i} = 2$ if b_{i+1} appears in row 2 of T, and $w_{b_i} = 1$ otherwise.

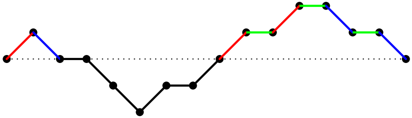
Tight 012-words and super Motzkin paths

Theorem

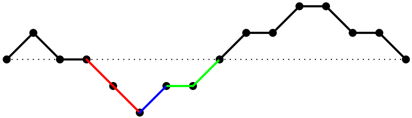
For any positive integer n, there is a bijection $\psi : \mathcal{SM}_n \mapsto \mathcal{W}_n$.



$\mapsto$ 0 2 0 0 0 2 1 1 0 1 0 1 2 1 2



$\mapsto$ 0 2 0 0 0 2 1 1 0 1 0 1 2 1 2



$\mapsto$ 0 2 0 0 0 2 1 1 0 1 0 1 2 1 2

Why do we define the tight 012-words?

1	3	5	6	8	11	12	19	20	21	22	23	26	30	31	32	36
2	4	10	15	18	25	27	34									
7																
9																
13																
14																
16																
17																
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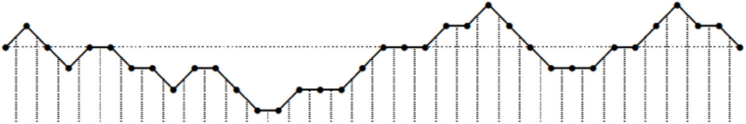
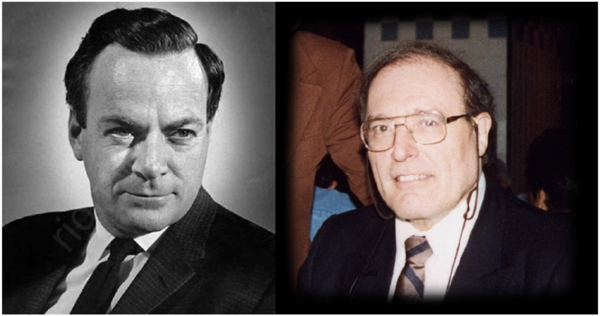


1	3	4	5	6	7	8	10	11	12	13	15	16	18	19	20	21	22	23	26	27	28	29	30	31	32	34	36
2	4	7	9	10	13	14	15	16	17	18	22	24	25	27	28	29	33	34	35								

Using the Feynman Method

Richard Feynman: You have to keep a dozen of your favorite problems constantly present in your mind. Every time you hear or read a new trick or a new result, test it against each of your twelve problems to see whether it helps. Every once in a while there will be a hit, and people will say, "How did he do it? He must be a genius!"



Gian-Carlo Rota, Ten Lessons I Wish I Had Been Taught, Notices of the AMS, 44(1), 1997.

Thank you!