A new construction of equiangular lines from integral lattices

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Basics

Definition

A collection X of lines in a Euclidean space is called equiangular if each pair from X intersects at a common angle.

$$|\langle v_i, v_j \rangle| = \alpha, \qquad \forall\, v_i, v_j \in X, v_i \neq v_j.$$

The elements in X are lines (through the origin), or unit vectors representing these lines.

Switching equivalent: replacing v by $-v$ for some subset of X.

Question: What is the maximum cardinality of an equiangular set in $\mathbb{R}^r$?

Related fields: Maximal packing, kissing number, maximum separation code, spherical design, tight frame, etc.

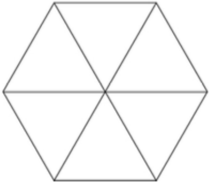

Figure: Maximum equiangular lines in $\mathbb{R}^2$: 3 lines through opposite vertices of a regular hexagon.

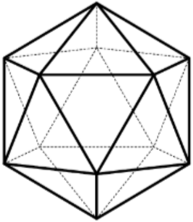

Figure: Maximum equiangular lines in $\mathbb{R}^3$: 6 lines through opposite vertices of the icosahedron.

Historical known results

Let $M(r)$ denote the maximum size of an equiangular line set in $\mathbb{R}^r$:

- Haantjes found $M(3) = M(4) = 6$ in 1948.
- Van Lint and Seidel found $M(5), M(6)$, and $M(7)$ in 1966.
- Lemmens and Seidel used linear-algebraic methods to determine $M(r)$ for most values of r in the region $8 \leq r \leq 23$ in 1973.

Table: Known bounds on $M(r)$ in small dimensions in 2008

r	$M(r)$	$1/\alpha$		r	$M(r)$	$1/\alpha$
2	3	2		17	48-50	5
3	6	$\sqrt{5}$		18	48-61	5
4	6	3; $\sqrt{5}$		19	72-76	5
5	10	3		20	90-96	5
6	16	3		21	126	5
7–13	28	3		22	176	5
14	28-30	3; 5		23	276	5
15	36	5		24–42	≥ 276	5; 7
16	40-42	5		43	≥ 344	7

Neumann Theorem

Why are the angles so special (1/3, 1/5, 1/7, etc.)?

Theorem (P. Neumann (1973))

If $\mathbb{R}^r$ contains M equiangular lines with angle α and if $M > 2r$, then $1/\alpha$ is an odd integer.

Improvements

Upper bounds for $M(r)$:

- Barg-Yu proved that $M(r) = 276$ for $24 \le r \le 41$ and $M(43) = 344$ (Contemp. Math., 2014).
- Azarija-Marc proved that $M(20) < 96$ (LAA 2018), and Yu proved that $M(19) < 76$ (arXiv 2015).
- Greaves-Koolen-Munemasa-Szöllősi proved that $M(14) < 30$ and $M(16) < 42$ (JCTA 2016).
- Greaves proved that $M(18) < 60$ (LAA 2018).
- Greaves-Yatsyna proved that $M(17) < 50$ (Math. of Comp., 2019).

Lower bounds for $M(r)$:

- Szöllősi proved that $M(18) \ge 54$. (DCGE 2019)
- L.-Yu proved that $M(18) \ge 56$. (2019+)

Current table

Table: Current known bounds on $M(r)$ in small dimensions

r	$M(r)$	$1/\alpha$		r	$M(r)$	$1/\alpha$
2	3	2		17	48-49	5
3	6	$\sqrt{5}$		18	56-59	5
4	6	3; $\sqrt{5}$		19	72-75	5
5	10	3		20	90-95	5
6	16	3		21	126	5
7–13	28	3		22	176	5
14	28-29	3; 5		23–41	276	5
15	36	5		42	≥ 276	5; 7
16	40-41	5		43	344	7

$\mathbb{R}^{14}$, angle $1/5$

Question: Is an equiangular set $X \subset \mathbb{R}^r$ maximal?

Example. $28 \leq M(14) \leq 29$.
There is a construction of 28 lines X in $\mathbb{R}^{14}$ with angle $1/5$.

Can we add one more line to X? Or can't we?

Example. 28 lines in $\mathbb{R}^{14}$ with angle $1/5$

From a $(7,3,1)$-design, we obtain an equiangular set X of 28 lines in $\mathbb{R}^{14}$ with angle $1/5$, as follows (cf. Tremain (2008)).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	○	○	●	●													○	●	●	○					○	●	○	●
2	○	●	○	●	○	○	●	●													○	●	●	○				
3					○	●	○	●	○	○	●	●													○	●	●	○
4	○	●	●	○					○	●	○	●	○	○	●	●												
5					○	●	●	○					○	●	○	●	○	○	●	●								
6									○	●	●	○					○	●	○	●	○	○	●	●				
7													○	●	●	○					○	●	○	●	○	○	●	●
8	⋆	⋆	⋆	⋆																								
9					⋆	⋆	⋆	⋆																				
10									⋆	⋆	⋆	⋆																
11													⋆	⋆	⋆	⋆												
12																	⋆	⋆	⋆	⋆								
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$$\circ = \sqrt{1/5}, \quad \bullet = -\sqrt{1/5}, \quad \star = \sqrt{2/5}$$

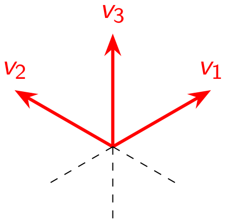
Seidel graphs

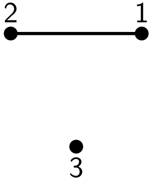
Let X be an equiangular set in $\mathbb{R}^r$ with angle α.

Definition

The Seidel graph of X is the simple graph whose vertex set consists of unit vectors from X and two vertices v_i, v_j are adjacent if and only if $\langle v_i, v_j \rangle = -\alpha < 0$.

Seidel graph, Adjacency matrix, Seidel matrix, Gram matrix

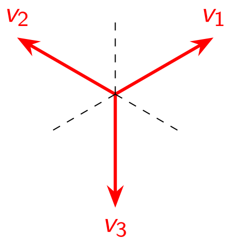


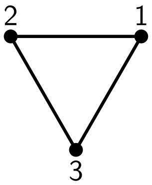


$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & -\alpha & \alpha \\ -\alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{bmatrix}$$

$$A = \frac{1}{2}(J - I - S); \qquad S = J - I - 2A; \qquad M = I + \alpha S.$$

Switching





$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & -\alpha & -\alpha \\ -\alpha & 1 & -\alpha \\ -\alpha & -\alpha & 1 \end{bmatrix}$$

$$A = \frac{1}{2}(J - I - S); \qquad S = J - I - 2A; \qquad M = I + \alpha S.$$

Link to integral lattices

Suppose that we have an equiangular set X of rank 14 with angle $1/5$.

Let M be the Gram matrix of X,
Let G be the complement of the Seidel graph of X.

$$M = I + \frac{1}{5} \cdot S(\overline{G}) \succeq 0 \quad \Rightarrow \quad \lambda_{\min}(S(\overline{G})) \geq -5$$
$$\Rightarrow \quad \lambda_{\max}(S(G)) \leq 5 \quad \Rightarrow \quad \lambda_{\min}(A(G)) \geq -3$$

(because $A = \frac{1}{2}(J - I - S)$.)

Link to integral lattices

$$\lambda_{\min}(A(G)) \geq -3 \quad \Rightarrow \quad T := A(G) + 3I \geq 0$$

Therefore T is:

1. a matrix whose diagonal entries are 3, and 0 or 1 off-diagonally;
2. the Gram matrix of some norm-3 vectors of rank 14.

Link to integral lattices

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Theorem (Conway-Sloane (1989))

Given an r-dimensional integral lattice, there is some k-dimensional odd unimodular lattice containing it, with $r \leq n + 3$.

So we look for unimodular odd integral lattices of ranks $14 \sim 17$.

The lattice $(E_7^2)^*$

The theta series for the lattice $(E_7^2)^*$ is

$$\theta(\tau) = 1 + 112q^{3/2} + 252q^2 + 3136q^3 + \cdots, \qquad q = \mathrm{e}^{\pi \mathrm{i} \tau},\ \mathrm{im}\tau > 0.$$

There are 3,136 norm-3 vectors in $(E_7^2)^*$; they are:

$$x = (v_1, v_2), \qquad v_i = \pm \text{ perm. of } \Big((\tfrac{1}{4})^6, (-\tfrac{3}{4})^2 \Big) \in \mathbb{R}^8.$$

Let G be a graph whose vertices are these norm-3 vectors, and $x \sim y$ in G if and only if $\langle x, y \rangle \in \{0, 1\}$.

A 28-clique of rank 14 is found in G. (Next slide: its Gram matrix)

The Gram matrix from the 28-clique

$$
\begin{pmatrix}
3&1&1&0&0&1&1&1&0&0&1&0&1&1&1&0&1&1&1&1&0&0&1&0&1&0&0&0\\
1&3&0&0&1&1&1&1&0&0&1&1&0&1&1&0&1&1&0&0&1&1&0&0&0&0&1&1\\
1&0&3&0&0&0&1&1&1&1&1&0&1&1&1&0&0&0&1&1&0&1&0&0&0&1&1&1\\
0&0&0&3&1&1&1&1&0&1&0&1&1&0&1&1&0&1&1&0&0&1&0&1&1&1&0&0\\
0&1&0&1&3&0&1&1&1&0&0&1&1&0&1&1&1&0&1&0&0&0&1&1&0&0&1&1\\
1&1&0&1&0&3&1&1&0&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0\\
1&1&1&1&1&1&3&1&1&1&0&0&0&0&1&0&1&1&0&0&0&1&1&0&0&1&0&0\\
1&1&1&1&1&1&1&3&0&0&0&1&1&1&0&1&0&0&0&1&0&0&0&0&0&1&1&1\\
0&0&1&0&1&0&1&0&3&1&1&0&1&1&0&1&1&0&0&0&1&1&1&1&1&0&0&1\\
0&0&1&1&0&1&1&0&1&3&1&1&0&0&1&0&1&0&0&1&0&1&0&1&1&1&0&1\\
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1&1&1&1&1&0&1&0&0&1&1&0&1&0&3&0&1&1&1&0&0&0&0&1&0&1&0&1\\
0&0&0&1&1&0&0&1&1&0&1&0&1&0&0&3&1&1&0&1&1&0&1&0&1&1&1&1\\
1&1&0&0&1&0&1&0&1&1&1&1&1&0&1&1&3&1&0&1&1&0&1&0&0&0&0&0\\
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0&1&1&0&1&1&0&1&0&0&1&1&0&0&0&1&0&0&1&1&1&1&1&0&0&1&3&1\\
0&1&1&0&1&0&0&1&1&1&1&0&0&1&1&1&0&0&0&1&0&0&0&1&1&1&1&3
\end{pmatrix}
$$

Discussion

- This graph is neither isomorphic nor switching equivalent to Tremain's construction.

  Note that both graphs are 12-regular, with the same Seidel eigenvalues $[3]^7, [7]^7, [-5]^{14}$.

  Tremain's construction has a large automorphism group of order 21,504, but ours only has the trivial automorphism. (We have to include these 28 vectors with their opposite for this assertion.)

  Munemasa uses `MAGMA` to ensure that Tremain's construction can also be found in $(E_7^2)^*$. Explicit construction?

Discussion

- Other unimodular integral lattices of ranks $14 \sim 17$?

  For example, $(A_{15})^*$ has 3,640 norm-3 vectors.

  There is a 35-clique of rank 15 found in the induced graph.

  This clique is strongly regular with parameters $(35, 18, 9, 9)$.

  By Waldron's theorem (2009), it can be added one more line to form an equiangular set of 36 lines of rank 15 (which hits the maximum).

  Two problems: Clique number? Any 29-clique of rank 14?

Thank you very much!