The A_α-spectra of graphs

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Outline

Outline

Basic notations

- Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The degree of the vertex v_i is denoted by d_i.
- Adjacency matrix: $A(G) = (a_{ij})_{n\times n}$,

$$a_{ij} = \begin{cases} 1 & \text{if} \quad v_i \sim v_j, \\ 0 & \text{if} \quad v_i \nsim v_j. \end{cases}$$

- Degree matrix: $D(G) = diag(d_1, d_2, \ldots, d_n)$
- Laplacian matrix: $L(G) = D(G) - A(G)$
- Signless Laplacian matrix: $Q(G) = D(G) + A(G)$

- Laplacian matrix and signless Laplacian matrix are all positive semi-definite, they contain the same eigenvalues if G is a bipartite graph.
- The Laplacian spectrum and signless Laplacian spectrum are given by the adjacency spectrum if G is a regular graph.

- In extremal spectral graph theory, there are many similar conclusions with respect to A-matrix and Q-matrix.

Graph type	Objective	Extremal graph
unicycle graphs	maximize the spectral radius / signless Laplaican spectral radius	same
bicyclic graphs	maximize the spectral radius / signless Laplaican spectral radius	same
graphs with given diameter	maximize the spectral radius /signless Laplaican spectral radius	same
graphs with given clique number	minimize the spectral radius /signless Laplaican spectral radius	same
...	...	...

- However, there are also a lot of differences between adjacency spectra and signless Laplacian spectra, and the research on $Q(G)$ has shown that it is a remarkable matrix, unique in many respects.

In order to study both similarities and differences between $A(G)$ and $Q(G)$, Nikiforov [1] introduced a new matrix $A_\alpha(G)$:

For a real number $\alpha \in [0,1]$, the A_α-matrix of G is

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G),$$

where $A(G)$ is the adjacency matrix and $D(G)$ is the degree diagonal matrix of G.

- A_α-eigenvalues: $\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \cdots \geq \lambda_n(A_\alpha(G))$
- A_α-spectral radius: $\lambda_1(A_\alpha(G))$

  - if $\alpha = 0$, then $A_\alpha(G) = A(G)$
  - if $\alpha = 1/2$, then $A_\alpha(G) = \frac{1}{2}Q(G)$
  - if $\alpha = 1$ then $A_\alpha(G) = D(G)$

[1] V. Nikiforov, *Merging the A- and Q-spectral theories*, Appl. Anal. Discrete Math. 11 (2017) 81-107.

For a graph G, the A_α-eigenvalues are increasing in α.

Theorem ([1] Nikiforov 2017)

Let $1 \geq \alpha \geq \beta \geq 0$. If G is a graph of order n, then

$$\lambda_k(A_\alpha(G)) \geq \lambda_k(A_\beta(G))$$

for any $k \in [n]$. If G is connected, then inequality is strict, unless $k = 1$ and G is regular.

Take $\alpha = 0, \frac{1}{2}, 1$, thus $\quad (A_0(G) = D(G), A_1(G) = A(G))$

$$\lambda_k(D(G)) \geq \lambda_k(A_{\frac{1}{2}}(G)) \geq \lambda_k(A(G)).$$

Take $k = 1$, thus

$$2\Delta(G) \geq q(G) \geq 2\rho(G),$$

where $\Delta(G)$ is the maximum degree, $q(G)$ is the signless Laplacian spectral radius and $\rho(G)$ is the spectral radius of G, respectively.

Positive semidefiniteness of $A_\alpha(G)$

Note that the signless Laplacian matrix is positive semidefinite, that is, $A_{\frac{1}{2}}(G)$ is positive semidefinite.

Theorem ([2] Nikiforov and Rojo 2017)

Let G be a graph. If $\alpha \geq 1/2$, then $A_\alpha(G)$ is positive semidefinite. If $\alpha > 1/2$ and G has no isolated vertices, then $A_\alpha(G)$ is positive definite.

[2] V. Nikiforov, O. Rojo, *A note on the positive semidefiniteness of $A_\alpha(G)$*, Linear Algebra Appl. 519 (2017) 156-163.

Outline

Theorem ([1] Nikiforov 2017)

Let $r \geq 2$ and G be an r-chromatic graph of order n.

(1) *If $\alpha < 1 - 1/r$, then $\lambda_1(A_\alpha(G)) \leq \lambda_1(A_\alpha(T_r(n)))$, with equality if and only if $G \cong T_r(n)$ (r-**partite Turán graph**)*

(2) *If $\alpha > 1 - 1/r$, then $\lambda_1(A_\alpha(G)) \leq \lambda_1(A_\alpha(S_{n,r-1}))$, with equality if and only if $G \cong S_{n,r-1}$ ($\mathbf{K_{r-1} \vee K^c_{n-r+1}}$).*

(3) *If $\alpha = 1 - 1/r$, then $\lambda_1(A_\alpha(G)) \leq (1 - 1/r)n$, with equality if and only if G is a complete r-partite graph.*

Theorem ([1] Nikiforov 2017)

Let $r \geq 2$ and G be a K_{r+1}-free graph of order n.

(1) *If $\alpha < 1 - 1/r$, then $\lambda_1(A_\alpha(G)) \leq \lambda_1(A_\alpha(T_r(n)))$, with equality if and only if $G \cong T_r(n)$.*

(2) *If $\alpha > 1 - 1/r$, then $\lambda_1(A_\alpha(G)) \leq \lambda_1(A_\alpha(S_{n,r-1}))$, with equality if and only if $G \cong S_{n,r-1}$.*

(3) *If $\alpha = 1 - 1/r$, then $\lambda_1(A_\alpha(G)) \leq (1 - 1/r)n$, with equality if and only if G is a complete r-partite graph.*

The techniques used here are partially from [*He, Jin and Zhang: Sharp bounds for the signless Laplacian spectral radius in terms of clique number, LAA 438 (2013) 3851-3861.*]

Theorem ([3] Nikiforov, Pastén, Rojo and Soto 2017)

If T is a tree of order n, then

$$\lambda_1(A_\alpha(T)) \leq \lambda_1(A_\alpha(K_{1,n-1})).$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Theorem ([3] Nikiforov, Pastén, Rojo and Soto 2017)

If G is a connected graph of order n, then

$$\lambda_1(A_\alpha(G)) \geq \lambda_1(A_\alpha(P_n)).$$

Equality holds if and only if $G \cong P_n$.

[3] V. Nikiforov, G. Pastén, O. Rojo, R.L. Soto, *On the $A_\alpha(G)$-spectra of trees*, LAA 520 (2017) 286-305.

Graph transformations on spectral radius

Let G be a connected graph and u, v be two distinct vertices of $V(G)$. Let $G_{p,q}(u, v)$ be the graph obtained by attaching the paths P_p to u and P_q to v. The following problem is inspired by the results of Li and Feng [5].

Problem ([4] Nikiforov and Rojo 2018)

For which connected graphs G the following statement is true: Let $\alpha \in [0, 1)$ and let u and v be non-adjacent vertices of G of degree at least 2. If $q \geq 1$ and $p \geq q + 2$, then $\rho_\alpha(G_{p,q}(u, v)) < \rho_\alpha(G_{p-1,q+1}(u, v))$.

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[4] V. Nikiforov, O. Rojo, On the α-index of graphs with pendent paths. Linear Algebra Appl. 550 (2018) 87-104.

[5] Q. Li, K. Feng, On the largest eigenvalue of graphs, Acta Math. Appl. Sin. 2 (1979) 167-175.

Let G be a connected graph and $u, v \in V(G)$ with $d(u), d(v) \geq 2$. Suppose that u and v is connected by a path $w_0(= v)w_1 \cdots w_{s-1}w_s(= u)$ where $d(w_i) = 2$ for $1 \leq i \leq s-1$. Let $G_{p,s,q}(u, v)$ be the graph obtained by attaching the paths P_p to u and P_q to v.

Theorem (Lin, Huang and Xue 2018)

Let $0 \leq \alpha < 1$. If $p - q \geq \max\{s + 1, 2\}$, then $\rho_\alpha(G_{p-1,s,q+1}(u, v)) > \rho_\alpha(G_{p,s,q}(u, v))$.

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[6] H. Lin, X. Huang, J. Xue, A note on the A_α-spectral radius of graphs, Linear Algebra Appl. 557 (2018) 430–437.

The above theorem implies that the following conjecture is true.

Conjecture ([4] Nikiforov and Rojo 2018)

Let $0 \leq \alpha < 1$ and $s = 0, 1$. If $p \geq q + 2$, then $\rho_\alpha(G_{p,s,q}(u,v)) < \rho_\alpha(G_{p-1,s,q+1}(u,v))$.

It needs to be noticed that, the above conjecture is independently confirmed by Guo and Zhou [7].

[7] H. Guo, B. Zhou, On the α-spectral radius of graphs, arXiv:1805.03456.

An internal path of G is a path P (or cycle) with vertices $v_1, v_2, \ldots, v_k$ (or $v_1 = v_k$) such that $d_G(v_1) \geq 3$, $d_G(v_2) \geq 3$ and $d_G(v_2) = \cdots = d_G(v_{k-1}) = 2$.

Theorem ([8] Li, Chen and Meng 2019)

Let G be a connected graph with $\alpha \in [0, 1)$ and uv be some edge on an internal path of G. Let G_{uv} denote the graph obtained from G by subdivision of edge uv into edges uw and wv. Then $\rho_\alpha(G_{uv}) < \rho_\alpha(G)$.

A simple application of this transformation is that

If G is a unicyclic graph, then $\rho_\alpha(G) \leq \rho_\alpha(K_{1,n-1} + e)$, and the equality holds iff $G \cong K_{1,n-1} + e$.

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[8] D. Li, Y. Chen, J. Meng, The A_α-spectral radius of trees and unicyclic graphs with given degree sequence, Appl. Math. Comput. 363 (2019) 124622.

Outline

The k-th largest eigenvalue

Theorem (Lin, Xue and Shu 2018)

Let G be a graph with n vertices. If $\alpha \geq 1/2$ and $e \notin E(G)$, then

$$\lambda_k(A_\alpha(G+e)) \geq \lambda_k(A_\alpha(G)).$$

▷ Using this theorem, we get an upper bound on the A_α-eigenvalue when $\alpha \geq 1/2$:

$$\lambda_k(A_\alpha(G)) \leq \lambda_k(A_\alpha(K_n)) = \alpha n - 1.$$

Problem

Which graphs satisfy $\lambda_k(A_\alpha(G)) = \alpha n - 1$?

[9] H. Lin, J. Xue, J. Shu, On the A_α-spectra of graphs, Linear Algebra Appl. 556 (2018) 210–219.

- When $\alpha = 1/2$, de Lima and Nikiforov [10] showed that $\lambda_k(A_{\frac{1}{2}}(G)) = \frac{1}{2}n - 1$ for $k \geq 2$ if and only if G has either k balanced bipartite components or $k+1$ bipartite components.

Theorem (Lin, Xue and Shu 2018)

Let G be a graph with n vertices and $\alpha > 1/2$. Then

$$\lambda_k(A_\alpha(G)) \leq \alpha n - 1$$

for $k \geq 2$, and equality holds if and only if G has k vertices of degree $n - 1$.

[10] L.S. de Lima, V. Nikiforov, On the second largest eigenvalue of the signless Laplacian, Linear Algebra Appl. 438 (2013) 1215–1222.

Chen, Li and Meng [11] showed that if $\Delta(G) < n-1$ and $1/2 < \alpha < 1$ then $\lambda_k(A_\alpha(G)) \leq (n-2)\alpha$.

Theorem ([11] Chen, Li and Meng 2019)

Let G be a graph of order n and $\Delta(G) < n-1$. If $1/2 < \alpha < 1$ then

$$\lambda_k(A_\alpha(G)) = (n-2)\alpha$$

if and only if $G \cong K_{\underbrace{2,\ldots,2}_{k-1}} \vee H$, where $\Delta(H) < n-2k-3$.

[11] Y. Chen, D. Li, J. Meng, On the second largest A_α-eigenvalues of graphs, Linear Algebra Appl. 580 (2019) 343–358.

The least eigenvalue

Lower bounds for the least eigenvalue:

- Let T be a tree of order $n \geq 2$. If $\frac{1}{2} < \alpha < 1$, then

$$\lambda_n(A_\alpha(T)) \geq 2\alpha - 1,$$

the equality holds if and only if $T \cong K_2$.

Theorem (Lin, Xue and Shu 2018)

Let G be a graph on n vertices with $\alpha > \frac{1}{2}$. If G has no isolated vertices, then

$$\lambda_n(A_\alpha(G)) \geq 2\alpha - 1,$$

the equality holds if and only if there is a component isomorphic to K_2.

Outline

Graphs determined by their A_α-spectra

The study of spectral characterizations of graphs has a long history. In [9, Concluding remarks], van Dam and Haemers proposed the following problem:

Problem

Which linear combination of $D(G)$, $A(G)$ and J gives the most DS graphs?

From [12, Table 1], van Dam and Haemers claimed that the signless Laplacian matrix $Q(G) = D(G) + A(G)$ would be a good candidate.

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[12] E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectrum? Linear Algebra Appl. 373 (2003) 241–272.

Definition

A graph G is said to be *determined by its A_α-spectrum* if all graphs having the same A_α-spectrum as G are isomorphic to G.

We focus on which graphs are determined by their A_α-spectra.

By enumerating the A_α-characteristic polynomials for all graphs on at most 10 vertices (see [13, Table 1]), it seems that A_α-spectra (especially, $\alpha > \frac{1}{2}$) are much more efficient than Q-spectra when we use them to distinguish graphs.

[13] X. Liu, S. Liu, On the A_α-characteristic polynomial of a graph, Linear Algebra Appl. 546 (2018) 274–288.

Proposition (Lin, Liu and Xue 2018)

Let $\alpha \in [0,1]$. If G and G' are two graphs with the same A_α-spectra, then we have the following statements:

(P1) $|V(G)| = |V(G')|$;

(P2) $|E(G)| = |E(G')|$;

(P3) If G is r-regular, then G' is r-regular;

Suppose that $d_1 \geq d_2 \geq \cdots \geq d_n$ and $d'_1 \geq d'_2 \geq \cdots \geq d'_n$ are the degree sequences of G and G', respectively. If $\alpha \in (0,1]$, then

(P4) $\sum_{1\leq i<j\leq n} d_i d_j = \sum_{1\leq i<j\leq n} d'_i d'_j$;

(P5) $\sum_{1\leq i\leq n} d_i^2 = \sum_{1\leq i\leq n} {d'}_i^2$.

[14] H. Lin X. Liu, J. Xue, Graphs determined by their A_α-spectra, Discrete Math. 342 (2019) 441–450.

- G^c: the complement of a graph G.

Theorem (Lin, Liu and Xue 2018)

The following graphs are determined by their A_α-spectra:

(a) the complete graph K_n;

(b) the star $K_{1,n-1}$ for $1/2 < \alpha \leq 1$;

(c) the path P_n for $0 \leq \alpha < 1$;

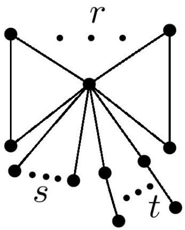
(d) the complement of a path P_n^c for $0 \leq \alpha < 1$;

(e) the union of cycles $\bigcup_{i=1}^{s} C_{n_i}$ for $0 \leq \alpha < 1$;

(f) $(\bigcup_{i=1}^{s} C_{n_i})^c$ for $0 \leq \alpha < 1$;

(g) $kK_2 \bigcup (n-2k)K_1$ where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $0 \leq \alpha \leq 1$;

(h) $(kK_2 \bigcup (n-2k)K_1)^c$ where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $0 \leq \alpha \leq 1$.



$F_{r,s,t}$

Theorem ([11] Chen, Li and Meng 2019)

If $1/2 < \alpha < 1$, then the firefly graph $F_{r,s,t}$ is determined by its A_α-spectrum.

More graphs determined by their A_α-spectra

▷ The join of a clique and an irregular graph:

Theorem (Lin, Liu and Xue 2018)

Let $m, n \geq 2$. Then $K_m \vee P_n$ is determined by its A_α-spectra for $\frac{1}{2} < \alpha < 1$.

▷ The join of a clique and a regular graph:

Theorem (Lin, Liu and Xue 2018)

Let G be a regular graph. If $1/2 < \alpha < 1$, then G is determined by its A_α-spectrum if and only if $G \vee K_m$ $(m \geq 2)$ is also determined by its A_α-spectrum.

Thank you for your attention!