Signed Countings of Type B and D Permutations and t, q-Euler numbers

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Outline

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1. History of the Problem
   - Signed countings on Permutations and Derangements
   - q-analogue of the signed counting identities

Definition

Let $\mathfrak{S}_n$ (resp. $\mathfrak{S}_n^*$) denote the set of permutations (resp. derangements) on $[n] := \{1, 2, \ldots, n\}$. We may express σ as $\sigma_1\sigma_2\ldots\sigma_n$ if $\sigma(i) = \sigma_i$ for all $1 \leq i \leq n$.
For any $\sigma \in \mathfrak{S}_n$, define the *descent*, *excedance* and *weak excedance* numbers of σ as follow:

- $\mathrm{des}(\sigma) := |\{i \in [n-1] : \sigma_i > \sigma_{i+1}\}|$
- $\mathrm{exc}(\sigma) := |\{i \in [n] : \sigma_i > i\}|$
- $\mathrm{wex}(\sigma) := |\{i \in [n] : \sigma_i \geq i\}|$

Example

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 3 & 5 & 4 & 8 & 1 & 2 & 6 \end{pmatrix} \in \mathfrak{S}_8$, we also write $\sigma = 73548126$.
Then $\mathrm{des}(\sigma) = 3$, $\mathrm{exc}(\sigma) = 4$, $\mathrm{wex}(\sigma) = 5$.

An elementary result states that des, exc, wex have the same distribution in $\mathfrak{S}_n$:

$$A_n(y) = \sum_{\sigma \in \mathfrak{S}_n} y^{\mathsf{des}(\sigma)+1} = \sum_{\sigma \in \mathfrak{S}_n} y^{\mathsf{exc}(\sigma)+1} = \sum_{\sigma \in \mathfrak{S}_n} y^{\mathsf{wex}(\sigma)}.$$

The polynomial $A_n(y)$ is called the *Eulerian polynomials* and $A_{n,k} = \#\{\sigma \in \mathfrak{S}_n | \mathsf{wex}(\sigma) = k\}$ is called the *Eulerian number*. We usually called the statistics which have the same distribution with these three the *Eulerian statistics*.

Euler numbers

Definition

The *Euler numbers* E_n are defined by

$$\tan x + \sec x = \sum_{n \geq 0} E_n \frac{x^n}{n!}$$

The numbers E_{2n} are called the *secant numbers* and the numbers E_{2n+1} are called the *tangent numbers*.

- The first few values are $1, 1, 1, 2, 5, 16, 61, 272, 1385, \ldots$.
- E_n counts the number of *alternating permutations* in $\mathfrak{S}_n$.
  i.e. $\mathsf{Alt}_n := \{\sigma \in \mathfrak{S}_n : \sigma_1 > \sigma_2 < \sigma_3 > \ldots \sigma_n\}$.
  e.g. $\mathsf{Alt}_3 = \{213, 312\}$,
  $\mathsf{Alt}_4 = \{2143, 3142, 3241, 4132, 4231\}$.

Signed counting identities

An interesting result occurs when we evaluate the Eulerean polynomials $A_n(y)$ at $y = -1$.

Theorem (Euler, 1755; Foata and Schützenberger, 1970)

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathsf{exc}(\sigma)} = \begin{cases} 0 & \text{, if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} E_n & \text{, if } n \text{ is odd.} \end{cases}$$

The other half shows up while we restrict our attention on the derangements in $\mathfrak{S}_n$!

Theorem (Roselle, 1968)

$$\sum_{\sigma \in \mathfrak{S}_n^*} (-1)^{\mathsf{exc}(\sigma)} = \begin{cases} (-1)^{\frac{n}{2}} E_n & \text{, if } n \text{ is even,} \\ 0 & \text{, if } n \text{ is odd.} \end{cases}$$

q-analogue of the signed counting identities

Definition (Crossing number of a permutation)

A *crossing* of a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ is a pair of (i, j) $(1 \le i < j \le n)$ such that

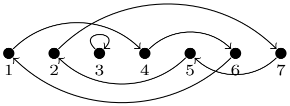
$$i < j \le \sigma_i < \sigma_j \qquad \text{or} \qquad \sigma_i < \sigma_j < i < j.$$

We denote by $\text{cro}(\sigma)$ the number of crossings in σ.

Crossings can be visualize via permutation diagram.

Example

Let $\sigma = 4736215$, $\text{cro}(\sigma) = 3$.



q-Eulerian numbers and polynomials

Lauren Williams (2005) introduce the notion of crossing along with this q-analogue of Eulerean numbers

$$A_{n,k}(q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{wex}(\sigma)=k}} q^{\mathsf{cro}(\sigma)}$$

$$A(y,q) = \sum_{k=1}^{n} A_{n,k}(q) y^k = \sum_{\sigma \in \mathfrak{S}_n} y^{\mathsf{wex}(\sigma)} q^{\mathsf{cro}(\sigma)}.$$

q-Euler numbers

Definition (Han, Randrianarivony, Zeng, 1999)

The *q-tangent numbers* $E_{2n+1}(q)$ and the *q-secant numbers* $E_{2n}(q)$ are defined by

$$\sum_{n=0}^{\infty} E_{2n+1}(q)z^n = \cfrac{1}{1 - \cfrac{[1]_q[2]_q z}{1 - \cfrac{[2]_q[3]_q z}{1 - \cfrac{[3]_q[4]_q z}{\ddots}}}}, \qquad \sum_{n=0}^{\infty} E_{2n}(q)z^n = \cfrac{1}{1 - \cfrac{[1]_q^2 z}{1 - \cfrac{[2]_q^2 z}{1 - \cfrac{[3]_q^2 z}{\ddots}}}}$$

The first few polynomials are $E_0(q) = E_1(q) = E_2(q) = 1$, $E_3(q) = 1 + q$, $E_4(q) = 2 + 2q + q^2$, $E_5(q) = 2 + 5q + 5q^2 + 3q^3 + q^4$.

q-Euler numbers

The polynomial $E_n(q)$ has a combinatorial interpretation (Chebikin,2008):

$$E_n(q) = \sum_{\sigma \in \mathsf{Alt}_n} q^{\text{31-2}(\sigma)}$$

where Alt_n is the set of alternating permutations of length n and 31-2$(\sigma) = \#\{(i,j) : i+1 < j, \sigma_{i+1} < \sigma_j < \sigma_i\}$.

Example

Alt_4	2143	3142	3241	4132	4231
31-2	0	1	0	2	1

$$\sum_{\sigma \in \mathsf{Alt}_4} q^{\text{31-2}(\sigma)} = 2 + 2q + q^2 = E_4(q)$$

q-analogue of the signed counting identities

Josuat-Vergès derived the following q-analogue of the signed counting identities.

Theorem (Josuat-Vergès, 2010)

For $n \geq 1$, we have

1. $$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{wex}(\pi)} q^{\mathsf{cro}(\pi)} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n+1}{2}} E_n(q) & \text{if } n \text{ is odd;} \end{cases}$$
2. $$\sum_{\pi \in \mathfrak{S}_n^*} (-\frac{1}{q})^{\mathsf{wex}(\pi)} q^{\mathsf{cro}(\pi)} = \begin{cases} \left(-\frac{1}{q}\right)^{\frac{n}{2}} E_n(q) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

- Note that permutations are the combinatorial model of the symmetric group $\mathfrak{S}_n$, which is merely the finite irreducible Coxeter group of type A.
- For type B and type D, there are combinatorial models similar to permutations. Fortunately, the notions we have mentioned, for instance wex, cro, also have type B analogs.
- One of our purpose in this work is to extend Josuat-Vergès' q-analogs of signed counting identities to type B and D.

Outline

Signed permutations

Definition

Let $[-n, n] := \{-n, -n+1, \ldots, -1, 1, 2, \ldots, n\}$.

- A *signed permutation* of $[n]$ is a bijection $\sigma : [-n, n] \to [-n, n]$ s.t. $\sigma(-i) = -\sigma(i)$ for all $i \in [-n, n]$. For convenience, we write $-i$ as $\bar{i}$. Sometimes we express σ as $\sigma_1\sigma_2 \ldots \sigma_n$, where $\sigma_i = \sigma(i)$ for $1 \leq i \leq n$. This is called the *window notation* of σ.
- An *even signed permutation* is a signed permutation with even number of negative entries in its window notation.

Denote B_n and D_n the set of signed permutations and even signed permutations of $[n]$ resp., and B_n^* (D_n^* resp.) the subset of B_n (D_n resp.) without fixed points.

For example, $B_2 = \{12, \bar{1}2, 1\bar{2}, \bar{1}\bar{2}, 21, 2\bar{1}, \bar{2}1, \bar{2}\bar{1}\}$, $D_2 = \{12, \bar{1}\bar{2}, 21, \bar{2}\bar{1}\}$, $B_2^* = \{\bar{1}\bar{2}, 21, 2\bar{1}, \bar{2}1, \bar{2}\bar{1}\}$, $D_2^* = \{\bar{1}\bar{2}, 21, \bar{2}\bar{1}\}$.

The type B and D analogous of signed countings we mainly consider are

$$\sum_{\sigma\in W}(-1)^{\lfloor\frac{\mathsf{fwex}(\sigma)}{2}\rfloor}t^{\mathsf{neg}(\sigma)}q^{\mathsf{cro}_B(\sigma)} \text{ and } \sum_{\sigma\in W^*}\left(-\frac{1}{q}\right)^{\lfloor\frac{\mathsf{fwex}(\sigma)}{2}\rfloor}t^{\mathsf{neg}(\sigma)}q^{\mathsf{cro}_B(\sigma)}$$

where $W = B_n,\ D_n$. So I will briefly introduce what the notations in the expressions mean and what the type B and D Euler numbers are and other related results.

The type B analogous of weak excedance we need here is the *flag weak excedance* of signed permutations.

Definition

For $\sigma \in B_n$, we define $\text{wex}(\sigma) = \#\{i \in [n] : \sigma_i \geq i\}$ and $\text{neg}(\sigma) = \#\{\sigma_i : i \in [n], \sigma_i < 0\}$. Then the *flag weak excedance number* is defined as

$$\text{fwex}(\sigma) = 2\text{wex}(\sigma) + \text{neg}(\sigma).$$

Example

$\sigma = \begin{pmatrix} \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 \\ 1 & \bar{3} & 4 & \bar{2} & \bar{5} & 5 & 2 & \bar{4} & 3 & \bar{1} \end{pmatrix} = 52\bar{4}3\bar{1} \in B_5.$

Then $\text{wex}(\sigma) = 2$ and $\text{neg}(\sigma) = 2 \Rightarrow \text{fwex}(\sigma) = 4 + 2 = 6.$

Crossing of type B

Definition (Corteel, Josuat-Vergès and Williams, 2011)

For $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in B_n$, a *crossing* of σ is a pair (i,j) with $i,j \geq 1$ such that

$$i < j \leq \sigma_i < \sigma_j \quad \text{or} \quad -i < j \leq -\sigma_i < \sigma_j \quad \text{or} \quad i > j > \sigma_i > \sigma_j.$$

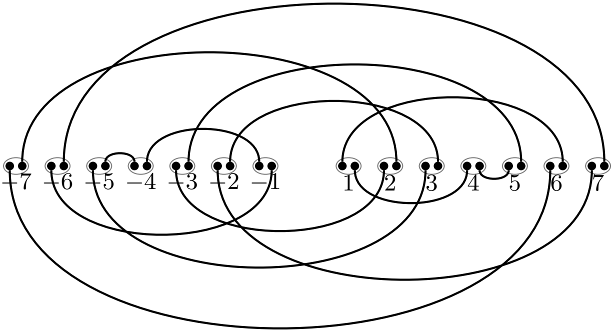
We write $\mathsf{cro_B}(\sigma)$ as the number of crossings of $\sigma \in B_n$.

Type B crossings can be visualized by *pignose diagram*.

Crossing of type B

Example

Let $\sigma = 6\bar{3}\bar{5}14\bar{7}\bar{2}$, the crossings are $(7,1), (3,1), (2,1)$ $(-i < j \leq -\sigma_i < \sigma_j)$ and $(4,2)$,$(4,3)$,$(7,2)$,$(7,3)$,$(7,6)$ $(i > j > \sigma_i > \sigma_j)$ so $\mathsf{cro_B}(\sigma) = 8$.



Type B analogous of q-Eulerian polynomials

Let $B_n(y,t,q) = \sum_{\sigma \in B_n} y^{\mathsf{fwex}(\sigma)} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro_B}(\sigma)}$. The first few values are

$$\begin{aligned} B_0(y,t,q) =& 1, \quad B_1(y,t,q) = y^2 + yt, \\ B_2(y,t,q) =& y^4 + (2t+tq)y^3 + (t^2q + t^2 + 1)y^2 + ty. \end{aligned}$$

Theorem (Corteel, Josuat-Vergès, Kim, 2013)

The continued fraction expansion for the generating fucntion of $B_n(y,t,q)$ *is*

$$\sum_{n \geq 0} B_n(y,t,q) x^n$$

$$= \cfrac{1}{1 - (y^2+yt)[1]_q z - \cfrac{(1+ytq)(y^2+yt)[1]_q{}^2 z^2}{1 - [(y^2+ytq)[2]_q + (1+ytq)[1]_q]z - \cfrac{(1+ytq^2)(y^2+ytq)[2]_q{}^2 z^2}{\ddots}}}$$

Gerneralized Euler numbers-Springer numbers

Springer (1971) defined the Springer number $K(W)$ for any Coxeter group W. In particular, $\mathfrak{S}_n$ is the irreducible Coxeter group of type A_{n-1} and $K(A_{n-1}) = K(\mathfrak{S}_n) = E_n$.

Definition

Let (W, S) be a Coxeter system, for any $w \in W$ the (right) descent set of w is defined to be

$$\mathrm{Des}(w) = \{s \in S : \ell(ws) < \ell(w)\}.$$

Let $J \subset S$ and $D_J = \{w \in W : \mathrm{Des}(w) = J\}$, then the *Springer number* of W is defined to be the cardinality of the largest descent classes

$$K(W) := \max_{J \subset S} |D_J|$$

Gerneralized Euler numbers-Springer numbers

Example

$W = \mathfrak{S}_3, \;\; S = \{s_1 = (12), s_2 = (23)\}$

Take $J = \{s_1\}$ or $\{s_2\}$,

$D_J = \{213, 312\}$ or $\{132, 231\}$.

Springer number $K(\mathfrak{S}_3) = 2 = E_3$

$w \in \mathfrak{S}_3$	$D(w)$
$123 = id$	$\emptyset$
$132 = s_2$	s_2
$213 = s_1$	s_1
$231 = s_1 s_2$	s_2
$312 = s_2 s_1$	s_1
$321 = s_1 s_2 s_1$	s_1, s_2

Gerneralized Euler numbers-Springer numbers

Denote $S_n := K(B_n)$ and $S_n^D := K(D_n)$.

n	0	1	2	3	4	5	6	...
E_n	1	1	1	2	5	16	61	...
S_n	1	1	3	11	57	361	2763	...
S_n^D	1	1	1	5	23	151	1141	...

Table: Springer number of type A, B and D

Combinatorial models of Springer numbers-Snakes

By describing Springer number geometrically in terms of Weyl chambers, Arnold (1992) showed Springer numbers of type A, B, D count various types of *snakes*.

Definition

Let $\sigma = \sigma_1 \ldots \sigma_n \in B_n$, then $\sigma \in B_n$ is a *snake* if $\sigma_1 > \sigma_2 < \sigma_3 > \ldots \sigma_n$.

1. Let $\mathcal{S}_n \subset B_n$ be the set of snakes of size n.
2. Let $\mathcal{S}_n^0 \subset \mathcal{S}_n$ be the subset consisting of the snakes σ with $\sigma_1 > 0$.
3. Let $\mathcal{S}_n^{00} \subset \mathcal{S}_n^0$ be the subset consisting of the snakes σ with $\sigma_1 > 0$ and $(-1)^n \sigma_n < 0$.

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Combinatorial models of Springer numbers-Snakes

For example, as $n = 2$

$$\mathcal{S}_2 = \{1\bar{2}, \bar{1}\bar{2}, 21, 2\bar{1}\}, \qquad \mathcal{S}_2^0 = \{1\bar{2}, 21, 2\bar{1}\}, \qquad \mathcal{S}_2^{00} = \{1\bar{2}, 2\bar{1}\}.$$

In general $|\mathcal{S}_n| = 2^n E_n$, $|\mathcal{S}_n^0| = S_n$, $|\mathcal{S}_n^{00}| = 2^{n-1} E_n$.

n	1	2	3	4	5	6	...
E_n	1	1	2	5	16	61	...
$2^n E_n$	2	4	16	80	512	3904	...
S_n	1	3	11	57	361	2763	...
$2^{n-1} E_n$	1	2	8	40	256	1952	...
S_n^D	1	1	5	23	151	1141	...

Links with derivatives of trigonometric functions

There is a surprising link between snakes and the derivatives of trigonometric fuctions. Hoffman (1999) and Josuat Vergès (2014) studied the polynomials P_n, Q_n and R_n defined as

$$\frac{d^n}{dx^n}\tan x = P_n(\tan x)$$
$$\frac{d^n}{dx^n}\sec x = Q_n(\tan x)\sec x$$
$$\frac{d^n}{dx^n}\sec^2 x = R_n(\tan x)\sec^2 x.$$

For examples,

$P_1(t) = 1 + t^2$, $\quad P_2(t) = 2t(1 + t^2) = 2t + 2t^3$, $\quad P_3(t) = 2 + 8t^2 + 6t^4$.
$Q_1(t) = t$, $\quad Q_2(t) = t^2 + (1 + t^2) = 1 + 2t^2$, $\quad Q_3(t) = 5t + 6t^3$.
$R_1(t) = 2t$, $\quad R_2(t) = 2 + 6t^2$, $\quad R_3(t) = 16t + 24t^3$.

Observe $P_n(1), Q_n(1), R_n(1)$.

Links with derivatives of trigonometric functions

Hoffman showed $P_n(1) = 2^n E_n$, $Q_n(1) = S_n$, $P_n(1) - Q_n(1) = S_n^D$.
Then Josuat-Vergès gave combinatorial interpretations to $P_n(t)$, $Q_n(t)$ and $R_n(t)$ in terms of number of *changes of sign* cs.

Theorem (Josuat-Vergès 2014)

For all $n \geq 0$, we have

$$P_n(t) = \sum_{\sigma \in \mathcal{S}_n} t^{\mathsf{cs}(\sigma)}, \qquad Q_n(t) = \sum_{\sigma \in \mathcal{S}_n^0} t^{\mathsf{cs}(\sigma)}, \qquad R_n(t) = \sum_{\sigma \in \mathcal{S}_{n+1}^{00}} t^{\mathsf{cs}(\sigma)},$$

where $\mathsf{cs}(\sigma) := \#\{i : \sigma_i \sigma_{i+1} < 0, 0 \leq i \leq n\}$ *with the following conventions*

- $\sigma_0 = -(n+1)$ *and* $\sigma_{n+1} = (-1)^n (n+1)$ *if* $\sigma \in \mathcal{S}_n$*;*
- $\sigma_0 = 0$ *and* $\sigma_{n+1} = (-1)^n (n+1)$ *if* $\sigma \in \mathcal{S}_n^0$*;*
- $\sigma_0 = 0$ *and* $\sigma_{n+1} = 0$ *if* $\sigma \in \mathcal{S}_n^{00}$.

Links with derivatives of trigonometric functions

Example

$\mathcal{S}_2 = \{(\bar{3})1\bar{2}(3), (\bar{3})\bar{1}\bar{2}(3), (\bar{3})21(3), (\bar{3})2\bar{1}(3)\}$, then

$$\sum_{\sigma \in \mathcal{S}_2} t^{\mathsf{cs}(\sigma)} = t^3 + t + t + t^3 = 2t + 2t^3 = P_2(t)$$

$\mathcal{S}_2^0 = \{(0)1\bar{2}(3), (0)21(3), (0)2\bar{1}(3)\}$, then

$$\sum_{\sigma \in \mathcal{S}_2^0} t^{\mathsf{cs}(\sigma)} = t^2 + 1 + t^2 = 1 + 2t^2 = Q_2(t).$$

$\mathcal{S}_2^{00} = \{(0)1\bar{2}(0), (0)2\bar{1}(0)\}$, then

$$\sum_{\sigma \in \mathcal{S}_2^{00}} t^{\mathsf{cs}(\sigma)} = 2t = R_1(t).$$

(t, q)-analogue of derivative polynomials

Definition (Josuat-Vergès,2014)

Define two polynomials of variable t, q

$$Q_n(t,q) = (D + UDU)^n 1, \quad R_n(t,q) = (D + DUU)^n 1,$$

where linear operator D,U is defined by

$$D(t^n) = [n]_q t^{n-1}, \quad U(t^n) = t^{n+1}.$$

Example

$Q_0(t,q) = 1$
$Q_1(t,q) = t$
$Q_2(t,q) = 1 + (1+q)t^2$
$Q_3(t,q) =$
$(2 + 2q + q^2)t + (1 + 2q + 2q^2 + q^3)t^3$

Example

$R_0(t,q) = 1$
$R_1(t,q) = (1+q)t$
$R_2(t,q) = (1+q) + (1 + 2q + 2q^2 + q^3)t^2$
$R_3(t,q) = (2 + 5q + 5q^2 + 3q^3 + q^4)t +$
$(1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6)t^3.$

(t, q)-analogue of derivative polynomials

Theorem (Josuat-Vergès, 2014)

The generating functions of $Q_n(t,q)$ *and* $R_n(t,q)$ *are*

$$\sum_{n\geq 0} Q_n(t,q) z^n = \cfrac{1}{1 - t[1]_q z - \cfrac{(1+t^2q)[1]_q^2 z^2}{1 - tq([1]_q + [2]_q)z - \cfrac{(1+t^2q^3)[2]_q^2 z^2}{1 - tq^2([2]_q + [3]_q)z - \cfrac{}{\ddots}}}}$$

and

$$\sum_{n\geq 0} R_n(t,q) z^n = \cfrac{1}{1 - t(1+q)[1]_q z - \cfrac{(1+t^2q^2)[1]_q[2]_q z^2}{1 - tq(1+q)[2]_q z - \cfrac{(1+t^2q^4)[2]_q[3]_q z^2}{1 - tq^2(1+q)[3]_q z - \cfrac{}{\ddots}}}}.$$

Outline

(t,q)-Signed Countings on B_n

Surprisingly, the signed countings of type B and D turn out to be related to the derivative polynomials $Q_n(t,q)$ and $R_n(t,q)$!

Theorem (Eu, Fu, Hsu, Liao 2018)

For $n \geq 1$, we have

1.
$$\sum_{\sigma\in B_n} (-1)^{\lfloor \frac{\mathrm{fwex}(\sigma)}{2} \rfloor} t^{\mathrm{neg}(\sigma)} q^{\mathrm{cro}_B(\sigma)} = \begin{cases} (-1)^{\frac{n}{2}}(t+1)R_{n-1}(t,q) & \text{, if } n \text{ is odd;} \\ (-1)^{\frac{n-1}{2}}(t-1)R_{n-1}(t,q) & \text{, if } n \text{ is even.} \end{cases}$$

2.
$$\sum_{\sigma\in B_n} (-1)^{\lceil \frac{\mathrm{fwex}(\sigma)}{2} \rceil} t^{\mathrm{neg}(\sigma)} q^{\mathrm{cro}_B(\sigma)} = \begin{cases} (-1)^{\frac{n}{2}}(t-1)R_{n-1}(t,q) & \text{if } n \text{ is even;} \\ (-1)^{\frac{n+1}{2}}(t+1)R_{n-1}(t,q) & \text{if } n \text{ is odd.} \end{cases}$$

(t,q)-Signed Countings on D_n

Corollary (Eu, Fu, Hsu, Liao 2018)

For $n \geq 1$, we have

$$\sum_{\sigma \in D_n} (-1)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \sum_{\sigma \in D_n} (-1)^{\lceil \frac{\mathsf{fwex}(\sigma)}{2} \rceil} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)}$$

$$= \begin{cases} (-1)^{\frac{n}{2}} t R_{n-1}(t,q) & \text{if } n \text{ is even,} \\ (-1)^{\frac{n+1}{2}} R_{n-1}(t,q) & \text{if } n \text{ is odd.} \end{cases}$$

(t,q)-Signed Countings on B_n^* and D_n^*

Theorem (Eu, Fu, Hsu, Liao 2018)

For $n \geq 1$*, we have*

1. $$\sum_{\sigma\in B_n^*} \left(-\frac{1}{q}\right)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \left(-\frac{1}{q}\right)^{\lfloor \frac{n}{2} \rfloor} Q_n(t,q).$$
2. $$\sum_{\sigma\in B_n^*} \left(-\frac{1}{q}\right)^{\lceil \frac{\mathsf{fwex}(\sigma)}{2} \rceil} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \left(-\frac{1}{q}\right)^{\lceil \frac{n}{2} \rceil} Q_n(t,q).$$

Corollary (Eu, Fu, Hsu, Liao 2018)

For $n \geq 1$*, we have*

$$\sum_{\sigma\in D_n^*} \left(-\frac{1}{q}\right)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \sum_{\sigma\in D_n^*} \left(-\frac{1}{q}\right)^{\lceil \frac{\mathsf{fwex}(\sigma)}{2} \rceil} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)}$$

$$= \begin{cases} \left(-\frac{1}{q}\right)^{\frac{n}{2}} Q_n(t,q) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Type B and D extension of Euler's result

Setting $t = 1$ and $q = 1$.

Corollary (Eu, Fu, Hsu, Liao 2018)

For $n \geq 1$, we have

1. $$\sum_{\sigma \in B_n} (-1)^{\lfloor \frac{\mathrm{fwex}(\sigma)}{2} \rfloor} = \begin{cases} (-1)^{\frac{n}{2}} 2^n E_n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

2. $$\sum_{\sigma \in B_n} (-1)^{\lceil \frac{\mathrm{fwex}(\sigma)}{2} \rceil} = \begin{cases} 0 & \text{if } n \text{ is even;} \\ (-1)^{\frac{n+1}{2}} 2^n E_n & \text{if } n \text{ is odd.} \end{cases}$$

3. $$\sum_{\sigma \in D_n} (-1)^{\lfloor \frac{\mathrm{fwex}(\sigma)}{2} \rfloor} = \sum_{\sigma \in D_n} (-1)^{\lceil \frac{\mathrm{fwex}(\sigma)}{2} \rceil} = (-1)^{\lfloor \frac{n+1}{2} \rfloor} 2^{n-1} E_n.$$

Type B and D extension of Roselle's result

Corollary (Eu, Fu, Hsu, Liao 2018)

For $n \geq 1$, we have

1. $$\sum_{\sigma \in B_n^*} (-1)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} = (-1)^{\lfloor \frac{n}{2} \rfloor} S_n.$$

2. $$\sum_{\sigma \in B_n^*} (-1)^{\lceil \frac{\mathsf{fwex}(\sigma)}{2} \rceil} = (-1)^{\lceil \frac{n}{2} \rceil} S_n.$$

3. $$\sum_{\sigma \in D_n^*} (-1)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} = \sum_{\sigma \in D_n^*} (-1)^{\lceil \frac{\mathsf{fwex}(\sigma)}{2} \rceil} = \begin{cases} (-1)^{\frac{n}{2}} S_n & \textit{if } n \textit{ is even}, \\ 0 & \textit{if } n \textit{ is odd}. \end{cases}$$

Springer numbers of type D

Recall that $P_n(1) - Q_n(1) = 2^n E_n - S_n = S_n^D$, this implies the following identities of Springer numbers of type D.

Corollary (Eu, Fu, Hsu, Liao 2018)

For $n \geq 1$, we have

1. $$\sum_{\sigma \in B_n - B_n^*} (-1)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} = \begin{cases} (-1)^{\frac{n}{2}} S_n^D & \text{if } n \text{ is even,} \\ (-1)^{\frac{n+1}{2}} S_n & \text{if } n \text{ is odd.} \end{cases}$$

2. $$\sum_{\sigma \in B_n - B_n^*} (-1)^{\lceil \frac{\mathsf{fwex}(\sigma)}{2} \rceil} = \begin{cases} (-1)^{\frac{n}{2}+1} S_n & \text{if } n \text{ is even,} \\ (-1)^{\frac{n+1}{2}} S_n^D & \text{if } n \text{ is odd.} \end{cases}$$

Signed Countings on type B derangements

The theorem below is one of our main results. I will briefly describe how we prove the theorem, then I will show all our signed countings results on type B and D.

Theorem (Eu, Fu, Hsu, Liao 2018)

For $n \geq 1$, we have

1. $$\sum_{\sigma \in B_n^*} \left(-\frac{1}{q}\right)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \left(-\frac{1}{q}\right)^{\lfloor \frac{n}{2} \rfloor} Q_n(t,q).$$

2. $$\sum_{\sigma \in B_n^*} \left(-\frac{1}{q}\right)^{\lceil \frac{\mathsf{fwex}(\sigma)}{2} \rceil} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \left(-\frac{1}{q}\right)^{\lceil \frac{n}{2} \rceil} Q_n(t,q).$$

where $\mathsf{neg}(\sigma) = |\{i \in [n] |\ \sigma_i < 0\}|$.

Signed Countings on derangements of type B and D

Example

B_2^*	fwex	neg	cro_B
$\bar{1}\bar{2}$	2	2	0
21	2	0	0
$\bar{2}1$	1	1	0
$2\bar{1}$	3	1	1
$\bar{2}\bar{1}$	2	2	1

$$\sum_{\sigma\in B_2^*} (-1)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_\mathsf{B}(\sigma)}$$

$$= \left(-\frac{1}{q}\right) t^2 + \left(-\frac{1}{q}\right) 1 + t + \left(-\frac{1}{q}\right) tq + \left(-\frac{1}{q}\right) t^2 q$$

$$= \left(-\frac{1}{q}\right) (t^2 + 1 - tq + tq + t^2 q)$$

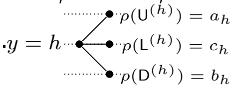
$$= \left(-\frac{1}{q}\right) (1 + (1+q)t^2)$$

$$= \left(-\frac{1}{q}\right) Q_2(t,q).$$

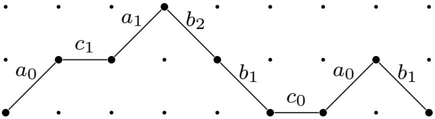
Weighted Motzkin paths

- A *Motzkin paths of length n* is a lattice path from $(0,0)$ to $(n,0)$ above the x-axis using steps $\mathsf{U} = (1,1)$, $\mathsf{L} = (1,0)$, $\mathsf{D} = (1,-1)$.

  e.g. length 3: 

- Assign each step a weight 

  For a weighted Motzkin path $\mathcal{P} = w_1 w_2 \dots w_n$, the weight of $\mathcal{P}$ is denoted by $\rho(\mathcal{P}) := \prod_{i=1}^{n} \rho(w_i)$.



Flajolet's Fundamental Lemma

Theorem (Flajolet's Fundamental Lemma 1980)

Let M_n be the set of weighted Motzkin paths of length n with weights given as before. Then the generating function of path weights in M_n for all n has the expansion

$$\sum_{n\geq 0}\sum_{\mathcal{P}\in M_n}\rho(\mathcal{P})z^n = \cfrac{1}{1-c_0z-\cfrac{a_0b_1z^2}{1-c_1z-\cfrac{a_1b_2z^2}{\cfrac{\ddots}{1-c_kz-\cfrac{a_kb_{k+1}z^2}{\ddots}}}}}$$

B_n and set $\mathcal{M}_n$ of corresponding paths

We had seen that generating functions of $B_n(y,t,q)$, $Q_n(t,q)$, $R_n(t,q)$ all have continued fraction expansions. For example, consider the GF of $B_n(y,t,q)$:

- $c_h = (y^2 + ytq^h)[h+1]_q + (1 + ytq^h)[h]_q$ $(h \geq 0)$,
  $a_h = (y^2 + ytq^h)[h]_q$ $(h \geq 0)$,
  $b_h = (1 + ytq^h)[h]_q$ $(h \geq 1)$.
- Set $\mathcal{M}_n$ of weighted bicolored Motzkin paths with weight function ρ defined as
  - $\rho(\mathsf{U}^{(h)}) \in \{y^2, y^2q, \ldots, y^2q^h\} \cup \{ytq^h, ytq^{h+1}, \ldots, ytq^{2h}\}$,
  - $\rho(\mathsf{L}^{(h)}) \in \{y^2, y^2q, \ldots, y^2q^h\} \cup \{ytq^h, ytq^{h+1}, \ldots, ytq^{2h}\}$,
  - $\rho(\mathsf{W}^{(h)}) \in \{1, q, \ldots, q^{h-1}\} \cup \{ytq^h, ytq^{h+1}, \ldots, ytq^{2h-1}\}$ for $h \geq 1$,
  - $\rho(\mathsf{D}^{(h+1)}) \in \{1, q, \ldots, q^h\} \cup \{ytq^{h+1}, ytq^{h+2}, \ldots, ytq^{2h+1}\}$.

Set of paths with weight $Q_n(t,q)$

Let $\mathcal{T}_n^*$ be the set of weighted bicolored Motzkin paths of length n containing no wavy level steps on the x-axis, with a weight function ρ such that for $h \geq 0$,

- $\rho(\mathsf{U}^{(h)}) \in \{1, q, \ldots, q^h\} \cup \{t^2 q^{2h+1}, t^2 q^{2h+2}, \ldots, t^2 q^{3h+1}\}$,
- $\rho(\mathsf{L}^{(h)}) \in \{tq^h, tq^{h+1}, \ldots, tq^{2h}\}$,
- $\rho(\mathsf{W}^{(h)}) \in \{tq^h, tq^{h+1}, \ldots, tq^{2h-1}\}$ for $h \geq 1$,
- $\rho(\mathsf{D}^{(h+1)}) \in \{1, q, \ldots, q^h\}$.

Then we have

$$\sum_{n\geq 0} \rho(\mathcal{T}_n^*) x^n = \sum_{n \geq 0} Q_n(t,q) x^n.$$

The case of B_n^*

Let $B_n^*(y,t,q) = \sum_{\sigma \in B_n^*} y^{\mathsf{fwex}(\sigma)} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro_B}(\sigma)}$. Observe that

$$B_n^*\left(\sqrt{\frac{-1}{q}}, t, q\right) = \sum_{\substack{\sigma \in B_n^* \\ 2 \mid \mathsf{fwex}(\sigma)}} \left(\frac{-1}{q}\right)^{\frac{\mathsf{fwex}(\sigma)}{2}} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)}$$

$$+ \sqrt{\frac{-1}{q}} \sum_{\substack{\sigma \in B_n \\ 2 \nmid \mathsf{fwex}(\sigma)}} \left(\frac{-1}{q}\right)^{\frac{\mathsf{fwex}(\sigma)-1}{2}} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)}.$$

It is easy to see that

$$\sum_{\sigma \in B_n^*} \left(\frac{-1}{q}\right)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro_B}(\sigma)} = \mathrm{Re}\left(B_n^*\left(\sqrt{\frac{-1}{q}}, t, q\right)\right) + \sqrt{q} \cdot \mathrm{Im}\left(B_n^*\left(\sqrt{\frac{-1}{q}}, t, q\right)\right)$$

and

$$\sum_{\sigma \in B_n^*} \left(\frac{-1}{q}\right)^{\lceil \frac{\mathsf{fwex}(\sigma)}{2} \rceil} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro_B}(\sigma)} = \mathrm{Re}\left(B_n^*\left(\sqrt{\frac{-1}{q}}, t, q\right)\right) - \sqrt{q} \cdot \mathrm{Im}\left(B_n^*\left(\sqrt{\frac{-1}{q}}, t, q\right)\right)$$

Paths corresponding to B_n^*

Theorem (Corteel, Josuat-Vergès, Kim 2013)

There is a bijection Γ between B_n and $\mathcal{M}_n$ such that

$$\sum_{\sigma \in B_n} y^{\mathsf{fwex}(\sigma)} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \rho(\mathcal{M}_n).$$

Γ restricts on B_n^* induces a bijection between B_n^* and subset $\mathcal{M}_n^* \subset \mathcal{M}_n$ whose weight scheme is the following:

- $\rho(\mathsf{U}^{(h)}) \in \{y^2, y^2q, \dots, y^2q^h\} \cup \{ytq^h, ytq^{h+1}, \dots, ytq^{2h}\}$,
- $\rho(\mathsf{L}^{(h)}) \in \{y^2, y^2q, \dots, y^2q^h\} \cup \{ytq^h, ytq^{h+1}, \dots, ytq^{2h}\}$ for $h \geq 1$ and $\rho(\mathsf{L}^{(0)}) \in \{yt\}$ for $h = 0$.
- $\rho(\mathsf{W}^{(h)}) \in \{1, q, \dots, q^{h-1}\} \cup \{ytq^h, ytq^{h+1}, \dots, ytq^{2h-1}\}$ for $h \geq 1$,
- $\rho(\mathsf{D}^{(h+1)}) \in \{1, q, \dots, q^h\} \cup \{ytq^{h+1}, ytq^{h+2}, \dots, ytq^{2h+1}\}$.

How do we prove the signed counting identities?

We construct an involution $\Psi_2 : \mathcal{M}_n^* \to \mathcal{M}_n^*$ that changes the weight of a path by the factor y^2q, with the following subset of $\mathcal{M}_n^*$ as the *fixed points*.

Let $\mathcal{G}_n \subset \mathcal{M}_n^*$ be the subset consisting of the paths satisfying the following conditions. For $h \geq 0$,

- $\rho(\mathsf{U}^{(h)}, \mathsf{D}^{(h+1)}) = (y^2q^a, q^b)$ or (ytq^{h+a}, ytq^{h+1+b}) for some $a, b \in \{0, 1, \ldots, h\}$, for any matching pair $(\mathsf{U}^{(h)}, \mathsf{D}^{(h+1)})$,
- $\rho(\mathsf{L}^{(h)}) \in \{ytq^h, ytq^{h+1}, \ldots, ytq^{2h}\}$,
- $\rho(\mathsf{W}^{(h)}) \in \{ytq^h, ytq^{h+1}, \ldots, ytq^{2h-1}\}$ for $h \geq 1$.

Comparing the weight with those of $\mathcal{T}_n^*$, we found

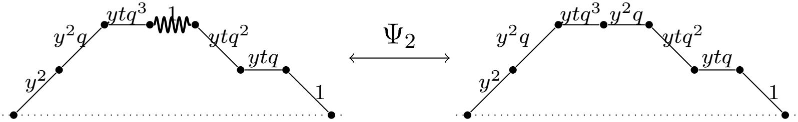
$$\rho(\mathcal{G}_n) = y^n \rho(\mathcal{T}_n^*) = y^n Q_n(t, q).$$

Involution $\Psi_2 : \mathcal{M}_n^* \longrightarrow \mathcal{M}_n^*$

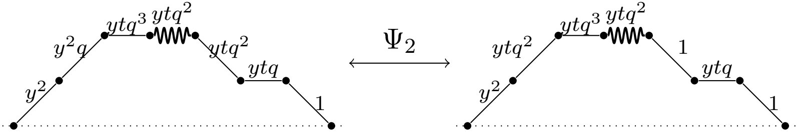
$\Psi_2 : \mathcal{M}_n^* \longrightarrow \mathcal{M}_n^*$, $\forall \mathcal{P} \in \mathcal{M}_n^*$,

1. If no step $\bullet\overset{y^2q^a}{\longrightarrow}\bullet$ (L) or $\bullet\overset{q^{a-1}}{\rightsquigarrow}\bullet$ (W) then go to (2). Otherwise, $\Psi_2(\mathcal{P}) := \mathcal{P}$ replace the 1st $\bullet\overset{y^2q^a}{\longrightarrow}\bullet$ ($\bullet\overset{q^{a-1}}{\rightsquigarrow}\bullet$, resp.) by $\bullet\overset{q^{a-1}}{\rightsquigarrow}\bullet$ ($\bullet\overset{y^2q^a}{\longrightarrow}\bullet$, resp.).
2. If no matching pair $\nearrow\overbrace{(y^2q^a, ytq^{h+1+b})}\searrow$ or $\nearrow\overbrace{(ytq^{h+a}, q^b)}\searrow$ ((U, D)) then go to (3). Otherwise, $\Psi_2(\mathcal{P}) := \mathcal{P}$ replace the 1st pair $\nearrow\overbrace{(y^2q^a, ytq^{h+1+b})}\searrow$ ($\nearrow\overbrace{(ytq^{h+a}, q^b)}\searrow$, resp.) by $\nearrow\overbrace{(ytq^{h+a}, q^b)}\searrow$ ($\nearrow\overbrace{(y^2q^a, ytq^{h+1+b})}\searrow$, resp.)
3. $\mathcal{P} \in \mathcal{G}_n$, $\Psi_2(\mathcal{P}) := \mathcal{P}$.

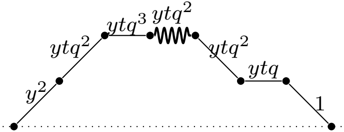
Example



Example



Example



To sum up, we have

$$\begin{aligned}
B_n^*\left(\sqrt{\frac{-1}{q}},t,q\right) &= \rho(\mathcal{M}_n^*)\Big|_{y=\sqrt{\frac{-1}{q}}} = \rho(\mathcal{G}_n)\Big|_{y=\sqrt{\frac{-1}{q}}} \\
&= y^n Q_n(t,q)\Big|_{y=\sqrt{\frac{-1}{q}}} \\
&= \begin{cases} \left(\frac{-1}{q}\right)^{\frac{n}{2}} Q_n(t,q) & \text{, if } n \text{ is even;} \\ \sqrt{\frac{-1}{q}}\left(\frac{-1}{q}\right)^{\frac{n-1}{2}} Q_n(t,q) & \text{, if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Hence

$$\sum_{\sigma\in B_n^*}\left(-\frac{1}{q}\right)^{\lfloor\frac{\mathsf{fwex}(\sigma)}{2}\rfloor} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \left(-\frac{1}{q}\right)^{\lfloor\frac{n}{2}\rfloor} Q_n(t,q),$$

$$\sum_{\sigma\in B_n^*}\left(-\frac{1}{q}\right)^{\lceil\frac{\mathsf{fwex}(\sigma)}{2}\rceil} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \left(-\frac{1}{q}\right)^{\lceil\frac{n}{2}\rceil} Q_n(t,q).$$

The case of D_n^*

Moreover, observe that

- Recall that $D_n^* \subset B_n^*$ consists of $\sigma \in B_n^*$ with even $\mathsf{neg}(\sigma)$.
- The involution Ψ_2 preserve the power of t.
- $\forall\ \mathcal{P} \in \mathcal{G}_n$, the parity of power of t is the same as that of n.

Therefore, we can easily derive the case of D_n^*.

Corollary (Eu, Fu, Hsu, Liao 2018)

For $n \geq 1$, we have

$$\sum_{\sigma \in D_n^*} \left(-\frac{1}{q}\right)^{\left\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \right\rfloor} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \sum_{\sigma \in D_n^*} \left(-\frac{1}{q}\right)^{\left\lceil \frac{\mathsf{fwex}(\sigma)}{2} \right\rceil} t^{\mathsf{neg}(\sigma)} q^{\mathsf{cro}_B(\sigma)}$$

$$= \begin{cases} \left(-\frac{1}{q}\right)^{\frac{n}{2}} Q_n(t,q) & \textit{if } n \textit{ is even,} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases}$$

Outline

- $\sigma \in \mathcal{S}_n^0$ (resp. $\mathcal{S}_n^{00}$) set $\sigma_0 := 0,\ \sigma_{n+1} := (-1)^n(n+1)$ (resp. $\sigma_0 = \sigma_{n+1} := 0$).
- Let $\mathfrak{S}_n^0$ (resp. $\mathfrak{S}_n^{00}$) be a copy of $\mathfrak{S}_n$ with convention $\sigma_0 := 0,\ \sigma_{n+1} := (-1)^n(n+1)$ (resp. $\sigma_0 = \sigma_{n+1} := 0$).
- Denote $|\sigma| := (|\sigma_0|)|\sigma_1| \dots |\sigma_n|(|\sigma_{n+1}|)$ for $\sigma \in \mathcal{S}_n^0, \mathcal{S}_n^{00}$.

Let $X(\sigma) =$Valley set with sign change of $|\sigma|$,
$Y(\sigma) =$DD and DA set of $|\sigma|$, $Z(\sigma) =$Peak set of $|\sigma|$.

Definition

Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n^0$ or $\mathfrak{S}_n^{00}$. For $1 \le i \le n$, we define

$$\text{13-2}(\pi, i) = \#\{j : 0 \le j < i-1 \text{ and } \pi_j < \pi_i < \pi_{j+1}\},$$
$$\text{2-31}(\pi, i) = \#\{j : i < j \le n \text{ and } \pi_j > \pi_i > \pi_{j+1}\}.$$

Let also $\text{2-31}(\pi) = \sum_{i=1}^{n} \text{2-31}(\pi, i)$.

The enumerator $Q_n(t,q)$ of $\mathcal{S}_n^0$

Theorem (Eu, Fu, Hsu, Liao 2018)

$$Q_n(t,q) = \sum_{\sigma \in \mathcal{S}_n^0} t^{\mathsf{cs}(\sigma)} q^{\text{2-31}(|\sigma|) + \mathsf{pat}_Q(\sigma)}.$$

where

$$\mathsf{pat}_Q(\sigma) = \sum_{j \in X(\sigma)} 2\big(\text{13-2}(|\sigma|, j) + \text{2-31}(|\sigma|, j)\big) - \#X(\sigma) + \sum_{j \in Y(\sigma)} \big(\text{13-2}(|\sigma|, j) + \text{2-31}(|\sigma|, j)\big).$$

Example

$\sigma \in \mathcal{S}_3^0$	$2(\#$cs X-blocks$)$	$-\#X(\sigma)$	$\#Y$-blocks	2-31$(\lvert\sigma\rvert)$	cs	
$(0)1\bar{2}3(\bar{4})$	0	0	0	0	3	t^3
$(0)1\bar{3}2(\bar{4})$	2×1	-1	0	0	3	t^3q
$(0)1\bar{3}\bar{2}(\bar{4})$	0	0	0	0	1	t
$(0)2\bar{1}3(\bar{4})$	2×1	-1	0	0	3	t^3q
$(0)213(\bar{4})$	0	0	0	0	1	t
$(0)2\bar{3}1(\bar{4})$	2×1	-1	1	1	3	t^3q^3
$(0)2\bar{3}\bar{1}(\bar{4})$	0	0	1	1	1	tq^2
$(0)312(\bar{4})$	0	0	1	0	1	tq
$(0)3\bar{1}2(\bar{4})$	2×1	-1	1	0	3	t^3q^2
$(0)3\bar{2}1(\bar{4})$	2×1	-1	1	0	3	t^3q^2
$(0)3\bar{2}\bar{1}(\bar{4})$	0	0	1	0	1	tq

$$\sum_{\sigma\in\mathcal{S}_3^0} t^{\mathsf{cs}(\sigma)} q^{\text{2-31}(|\sigma|)+\mathsf{pat}_Q(\sigma)}$$

$$=(2+2q+q^2)t+(1+2q+2q^2+q^3)t^3$$

The enumerator $R_n(t,q)$ of $\mathcal{S}_{n+1}^{00}$.

Theorem (Eu, Fu, Hsu, Liao 2018)

$$R_n(t,q) = \sum_{\sigma \in \mathcal{S}_{n+1}^{00}} t^{\mathsf{cs}(\sigma)} q^{\mathsf{2\text{-}31}(|\sigma|) + \mathsf{pat}_R(\sigma) - n - 1}.$$

where

$$\mathsf{pat}_R(\sigma) = \sum_{j \in X(\sigma)} 2\big(\mathsf{13\text{-}2}(|\sigma|, j) + \mathsf{2\text{-}31}(|\sigma|, j) - 1\big) + \sum_{j \in Y(\sigma)} \big(\mathsf{13\text{-}2}(|\sigma|, j) + \mathsf{2\text{-}31}(|\sigma|, j)\big) + \#Z(\sigma).$$

Outline

Discussion

- There are additional signed countings results involving type D Springer number S_n^D without (t,q)-extension.
  Any (t,q)-extension?
  The set of snakes of type D is defined to be

  $$\mathcal{S}_n^D = \{\sigma \in D_n |\ \sigma_1 + \sigma_2 < 0 \text{ and } \sigma_1 > \sigma_2 < \sigma_3 > \ldots\}.$$

  Any relations between the distribution of signed changing on $\mathcal{S}_n^D$ and $\sum_{B_n - B_n^*} (-1)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} t^{\mathsf{neg}(\sigma)}$?

Discussion

- Recall that in type A there is a notion in some sense dual to crossings which is called *nestings*. The joint distribution of crossing number and nesting number are symmetric in $\mathfrak{S}_n$. A type B analogous result had been proved in 2011 by Hamdi.

  A *type B nesting* of σ is defined as (i,j) with $i,j \geq 1$ satisfying

  $$i < j \leq \sigma_j < \sigma_i \text{ or } -i < j \leq \sigma_j < -\sigma_i \text{ or } j > i > \sigma_i > \sigma_j.$$

  Denote $\mathrm{nest_B}(\sigma)$ the number of nestings in σ.
  Consider the (p,q)-derivative $D_{p,q}$

  $$(D_{p,q}f)(t) := \frac{f(pt) - f(qt)}{(p-q)t},$$

  then $D_{p,q}(t^n) = [n]_{p,q} t^{n-1}$. Similarly, we may define $Q_n(t,p,q)$ and $R_n(t,p,q)$.

Discussion

Conjecture

For $n \geq 1$, we have

1. $$\sum_{\sigma \in B_n} (-1)^{\lfloor \frac{\mathsf{fwex}(\sigma)}{2} \rfloor} t^{\mathsf{neg}(\sigma)} p^{\mathsf{nest}_B(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \begin{cases} (-1)^{\frac{n}{2}} (t+1) R_{n-1}(t,p,q) \\ \text{, if } n \text{ is odd;} \\ (-1)^{\frac{n-1}{2}} (t-1) R_{n-1}(t,p,q) \\ \text{, if } n \text{ is even.} \end{cases}$$

2. $$\sum_{\sigma \in B_n} (-1)^{\lceil \frac{\mathsf{fwex}(\sigma)}{2} \rceil} t^{\mathsf{neg}(\sigma)} p^{\mathsf{nest}_B(\sigma)} q^{\mathsf{cro}_B(\sigma)} = \begin{cases} (-1)^{\frac{n}{2}} (t-1) R_{n-1}(t,p,q) \\ \text{if } n \text{ is even;} \\ (-1)^{\frac{n+1}{2}} (t+1) R_{n-1}(t,p,q) \\ \text{if } n \text{ is odd.} \end{cases}.$$

If the conjecture holds, naturally we have the derivation of type D from the conjecture. However, we haven't formulate the conjecture of similar signed counting identities for set B_n^* of type B derangements.

Discussion

- Generalize our signed counting results to colored permutations $\mathbb{Z}_r \wr \mathfrak{S}_n$. The results without parameter t,q has been prove by Athanasiadis as a byproduct of studying the γ-nonnegativity on Eulerian polynomial of $\mathbb{Z}_r \wr \mathfrak{S}_n$.

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Thanks for your attention.