Subgraphs in edge-colored graphs

Guanghui Wang

Shandong University, 山东大学

joint work with Cheng, Yang et al.
Taiwan, 2019. 8

Contents

Terminology and notation

- Edge-colored graph: a graph G with an edge coloring c of G
- Properly (edge) colored subgraph (or simply PC subgraph): no two adjacent edges have the same color
- Rainbow subgraph: no two edges have the same color
- Monochromatic subgraph: all edges have the same color

Contents

Under certain conditions, G has property $\mathcal{P}$ ("Hamiltonian"). Determine the robustness of G with respect to $\mathcal{P}$.

- Dirac graph has many Hamiltonian cycles.
- Maker can win a (Maker-Breaker) Hamiltonian game.
- Resilience.
- Compatible Hamiltonian cycle.

B. Sudakov, Robustness of graph properties, in: Surveys in Combinatorics 2017, Cambridge University Press, 2017, 372-408.

Theorem (Krivelevich, Lee, Sudakov, 2017)

There exists a constant $\mu = 10^{-16} > 0$ such that the following holds for large enough n. For every n-vertex locally μn-bounded edge colored Dirac graph G there exists a properly colored Hamiltonian cycle.

M. Krivelevich, C. Lee and B. Sudakov, Compatible Hamilton cycles in Dirac graphs, Combinatorica 37(2017), 697-732.

Theorem (Cousin, Perarnau, 2018+)

There exists a constant $\mu > 0$ such that the following holds for large enough n. If G is a Dirac graph on n vertices, then any μn-bounded coloring of $E(G)$ contains a rainbow Hamiltonian cycle.

M. Cousin and G. Perarnau, A Rainbow Dirac's theorem, https://arxiv.org/pdf/1809.06392.pdf

Contents

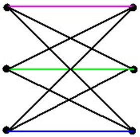
Rainbow matchings

Conjecture (Ryser, 1967, Brualdi-Stein, 1975)

Every latin square of order n has a partial latin transversal of size at least $n-1$. Moreover, every latin square of odd order has a Latin transversal.

latin transversal $\iff$ rainbow perfect matching.

1	2	3
2	3	1
3	1	2



Conjecture (Aharoni and Berger, 2009)

In every proper edge-colouring of a bipartite multigraph by n colors with at least $n+1$ edges of each color, there is a rainbow matching using every color.

Theorem (Pokrovskiy, 2018)

If there are at least $n+o(n)$ edges of each color in a proper n-edge-coloring of a bipartite multigraph, then there is a rainbow matching using every color.

- R. Aharoni, E. Berger, Rainbow matchings in r-partite r-graphs, Electron. J. Combin. 16, 2009.
- A. Pokrovskiy, An approximate version of a conjecture of Aharoni and Berger, Advances in Mathematics, 333: 1197-1241, 2018.

Conjecture (Barát, Gyárfás, Sárközy, 2017)

Any $2n$ matchings of size n on the same vertex set have a rainbow matching of size n.

Theorem (Aharoni, Berger, Chudnovsky, Howard, Seymour, 2019)

Any $3n-2$ matchings of size n on the same vertex set have a rainbow matching of size n.

J. Barát, A. Gyárfás, G. N. Sárközy, Rainbow matchings in bipartite multigraphs, Period. Math. Hungar. 74 (1): 108-111, 2017.

R. Aharoni, F. Berger, M. Chudnovsky, D. Howard, P. Seymour. Large rainbow matchings in general graphs, Euro. J. Combin., 79: 222-227, 2019.

Theorem (Aharoni, De la Maza, Montejano, Šámal, 2018)

Let G_1, G_2, G_3 be graphs on a common vertex set of size n. If $|E(G_i)| > \frac{1+\tau^2}{4}n^2$ for $1 \leq i \leq 3$ where $\tau = \frac{4-\sqrt{7}}{9}$, then there exists a rainbow triangle.

They also showed that τ^2 cannot be replaced by any $\varepsilon > 0$ with $\varepsilon < \tau^2$, this asymptotically answered a question of Mubayi [1] about a three-colored version of Mantel's theorem.

R. Aharoni, S. G. H. de la Maza, A. Montejano, R. Šámal, A rainbow version of Mantel's Theorem, arXiv:1812.11872.

A. Diwan, D. Mubayi, Turáns theorem with colors. http://www.math.cmu.edu/ mubayi/papers/webturan.pdf, 2006.

Question (Mubayi, 2006)

For every positive integer r, what is the smallest δ_r such that whenever $G_1, ..., G_{\binom{r}{2}}$ are graphs on a common set of n vertices with $|E(G_i)| \geq \delta_r n^2$ for every $1 \leq i \leq \binom{r}{2}$ there exists a rainbow K_r.

A. Diwan, D. Mubayi, Turáns theorem with colors. http://www.math.cmu.edu/ mubayi/papers/webturan.pdf, 2006.

Rainbow Hamiltonian cycles

Hamiltonian cycle (Dirac's Theorem)

Every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq \frac{n}{2}$ contains a Hamiltonian cycle.

G. A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc., 2(3): 69-81, 1952.

Rainbow Hamiltonian cycles

Theorem (Dirac)

Every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq \frac{n}{2}$ contains a Hamiltonian cycle.

Conjecture (Aharoni et al., 2018)

Given graphs $G_1, \cdots, G_n$ on the same vertex set of size n, each having degrees at least $n/2$, there exists a rainbow Hamiltonian cycle: a cycle with edge-set $\{e_1, \cdots, e_n\}$ such that $e_i \in E(G_i)$ for $i = 1, \cdots, n$.

R. Aharoni, S. G. H. de la Maza, A. Montejano, R. Šámal, A rainbow version of Mantel's Theorem, arXiv:1812.11872.

Our contribution

Theorem (Cheng, W, Zhao, 2019+)

For every $\varepsilon > 0$, there exists an integer $N > 0$, such that when $n > N$ for any graphs $G_1, ..., G_n$ on the same vertex set of size n, each graph having minimum degree at least $(\frac{1}{2} + \varepsilon)n$, there exists a rainbow Hamiltonian cycle.

Sketch of proof

We will use the "absorbing technique" introduced by Rödl, Ruciński, and Szemerédi.

V. Rődl, A. Ruciński, and E. Szemerédi, An approximate Dirac-type theorem for k-uniform hypergraphs, Combinatorica, 28 (2008), pp. 229 – 260.

A. Lo, An edge-colored version of Dirac's theorem, SIAM J. Discrete Math., 28(1): 18-36, 2014.

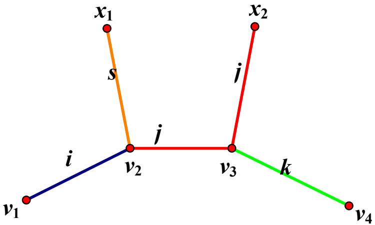
Rainbow Hamiltonian path

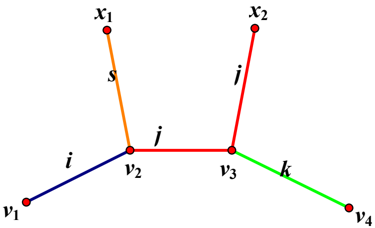
Lemma1

Given graphs $G_1, \cdots, G_n$ on the same vertex set of size n, each having degrees at least $n/2$, there exists a rainbow Hamiltonian path: a path with edge-set $\{e_1, \cdots, e_n\}$ such that $e_i \in E(G_i)$ for $i = 1, \cdots, n$.

For each pair of vertices x_1, x_2 and four colors $\{s, i, j, k\}$, we define the absorbing paths set $A_{s,i,j,k}(x_1, x_2)$ to be a family of rainbow 3-paths satisfying:

- $P = v_1v_2v_3v_4$ where $\{v_1, v_2, v_3, v_4\} \cap \{x_1, x_2\} = \emptyset$;
- v_1v_2, v_2v_3, v_3v_4 are colored by i, j, k;
- $c(x_1v_2) = s$ and $c(x_2v_3) = j$.





Lemma 2

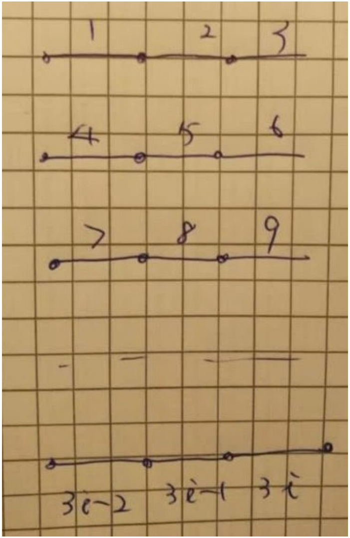
For each pair of vertices (x_1, x_2) and four distinct colors s, i, j, k, we have $|A_{s,i,j,k}(x_1, x_2)| \geq \frac{\varepsilon n^4}{8}$ when n is sufficiently large.

Lemma 3

We can find a "absorbing set" (absorbing paths, probabilistic method), which is :

- "not big".
- containing many absorbing paths.

- randomly and uniformly select rainbow paths
- "cleaning"



Lemma 4

Let $X = t_1 + ... + t_n$ where t_i are independent boolean random variables. Then for any $\varepsilon > 0$,

$$P(|X - E(X)| \geq \varepsilon E(X)) \leq 2e^{-min(\frac{\varepsilon^2}{4}, \frac{\varepsilon}{2})E(X)}.$$

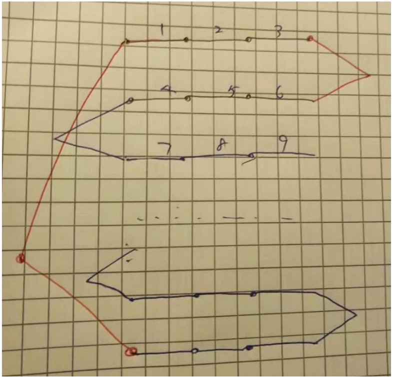
Especially, when $\varepsilon = \frac{1}{2}$, we conclude that

$$P(\frac{1}{2}E(X) < X < \frac{3}{2}E(X)) \geq 1 - 2e^{-\frac{E(X)}{16}}.$$

Lemma 5 (connecting lemma)

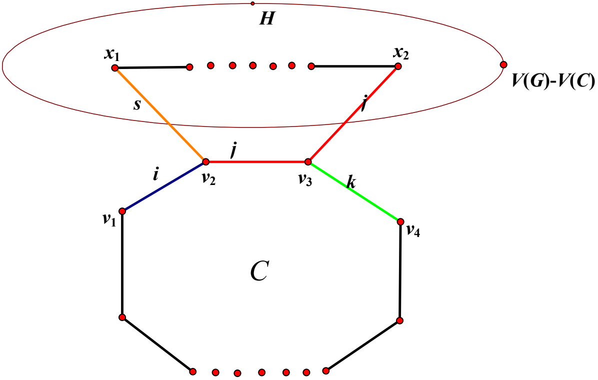
$\exists$ a rainbow cycle C (Absorbing cycle) ($|C| \leq \frac{\varepsilon n}{10^5}$) such that $\forall P \subseteq V(G) - V(C)$ ($|P| \geq 4$, Color(P)$\cap$ Color (C)=$\emptyset$) and a color s not used in C and P, $\exists$ a rainbow cycle $C^{'}$ such that

- $V(C') = V(C) \cup V(P)$;
- Color$\{(P \cup C) \cup \{s\}\}$=Color$(C')$.



"absorbing"

- path cover lemma
- absorbing



Contents

Properly colored cycles

Hamiltonian cycle (Dirac's Theorem)

Every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq \frac{n}{2}$ contains a Hamiltonian cycle.

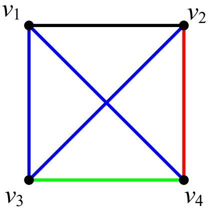
El-Zahár's Conjecture

Let t be a positive integer, and let G be a graph of order $n = \sum_{i=1}^{t} n_i$, where $n_i \geq 3$ for $1 \leq i \leq t$. If $\delta(G) \geq \sum_{i=1}^{t} \lceil n_i/2 \rceil$, then G contains t cycles (2-factor) of lengths $n_1, n_2, \ldots, n_t$, respectively.

- G. A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc., 2(3): 69-81, 1952.
- M. H. El-Zahár, On circuits in graphs, Discrete Math., 50: 227-230, 1984.

color degree

- $d^c(v)$: the number of distinct colors of edges incident to v, called the color degree
- $\delta^c(G)$: the minimum $d^c(v)$ over all vertices v in G, called the minimum color degree of an edge-colored graph G



$d^c(v_1)=d^c(v_3)=2$

$d^c(v_2)=d^c(v_4)=3$

A natural generalization of Dirac's theorem is to determine the minimum color degree threshold for the existence of a properly colored Hamiltonian cycle or properly colored 2-factors.

PC Hamiltonian cycles (Edge colored version of Dirac's Theorem?)

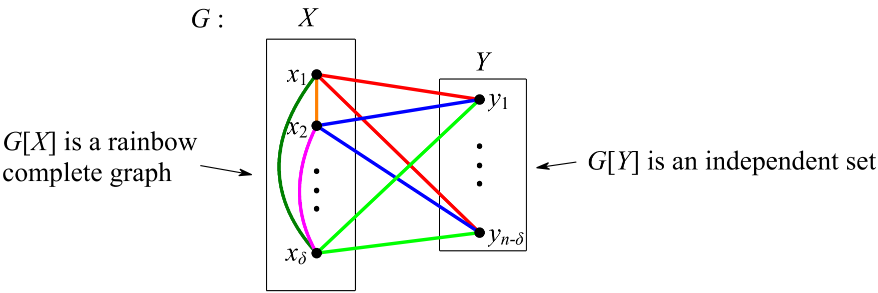
Every graph G on $n \geq 3$ vertices with minimum color degree $\delta^c(G) \geq \frac{n}{2}?$ contains a properly colored Hamiltonian cycle.

Theorems (Lo, 2014)

- For any $\varepsilon > 0$, there exists an integer n_0 such that every edge-colored graph G with $\delta^c(G) \geq (2/3 + \varepsilon)|G|$ and $|G| \geq n_0$ contains a properly colored Hamiltonian cycle.

A. Lo, An edge-colored version of Dirac's theorem, SIAM J. Discrete Math., 28(1): 18-36, 2014.

Example



$$|G| = n,\ X \cap Y = \emptyset,\ \delta < 2n/3$$

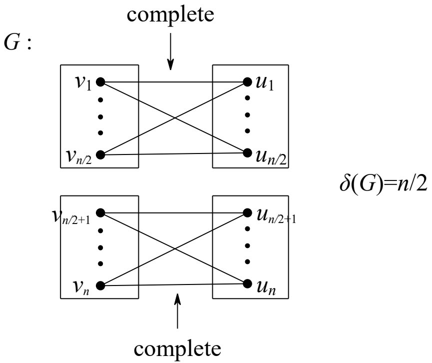
Our Contribution

Theorem (Cheng, Hu, W, 2019+)

For any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N(\varepsilon, t)$ such that every edge-colored graph G with $\delta^c(G) \geq (2/3 + \varepsilon)|G|$ and $|G| > N$ contains every properly colored 2-factor with exactly t components.

Moon and Moser's Theorem

Every balanced bipartite graph G on $2n$ vertices with minimum degree $\delta(G) > n/2$ is Hamiltonian.

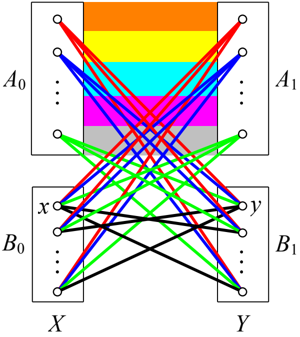


J. Moon, L. Moser, On hamiltonian bipartite graphs, Israel J. Math., 1: 163-165, 1963.

Our contribution

Theorem (Cheng, Hu, W, 2019+)

Every edge-colored balanced bipartite graph $G = (X, Y)$ of order $2n$ with $\delta^c(G) \geq 2n/3 + 1$ contains a PC 2-factor.



Theorem (Cheng, Hu, W, 2019+)

For any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N(\varepsilon, t)$ such that every edge-colored balanced bipartite graph G of order $2n$ with $\delta^c(G) \geq (2/3 + \varepsilon)n$ and $2n > N$ contains every PC even 2-factor with exactly t components.

Corollary

For any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that every edge-colored balanced bipartite graph G of order $2n$ with $\delta^c(G) \geq (2/3 + \varepsilon)n$ and $n > N$ contains a PC Hamiltonian cycle.

Theorem (Cheng, Hu, W, 2019+)

For any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N(\varepsilon, t)$ such that every edge-colored balanced bipartite graph G of order $2n$ with $\delta^c(G) \geq (2/3 + \varepsilon)n$ and $2n > N$ contains every PC even 2-factor with exactly t components.

Corollary

For any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that every edge-colored balanced bipartite graph G of order $2n$ with $\delta^c(G) \geq (2/3 + \varepsilon)n$ and $n > N$ contains a PC Hamiltonian cycle.

1-factors in digraphs

Theorem (Nash-Williams)

Every digraph D with $\delta^0(D) \geq |D|/2$ has a directed Hamiltonian cycle.

Theorem (Cheng, W, 2018+)

For any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N(\varepsilon, t)$ such that every digraph D with $\delta^0(D) \geq (1/2 + \varepsilon)|D|$ and $|D| > N$ contains every 1-factor with exactly t components.

Further research

- pc triangle factors
- arbitrary pc 2-factors
- Ramsey-Turan type (independence number sublinear)
- directed triangle factors in digraphs
- arbitrary 1-factors in digraphs

Contents

Theorem (Stiebitz, 1996)

Every G with minimum degree $\delta(G) \geq s+t+1$ has a partition (A, B) such that $\delta(G[A]) \geq s$ and $\delta(G[B]) \geq t$.

tight for K_{s+t+1}. In fact, he got a more general result.

Theorem (Stiebitz, 1996)

If $d_G(x) \geq s(x)+t(x)+1$ for $\forall x \in V(G)$, where s, t are two functions from $V(G)$ to $\mathbb{N}$, then G has a partition (A, B) such that $d_A(X) \geq s(x)$ for $\forall x \in A$ and $d_B(x) \geq t(x)$ for $\forall x \in B$.

- L. Lovasz, On decomposition of graphs, Studia Sci. Math. Hungar. 1 1966 237-238.
- M. Stiebitz, Decomposing graphs under degree constraints, J. Graph Theory 23 (1996), no. 3, 321-324.

Minimum outdegree

Problem

For integers s, t, does there exist a smallest value $f(s,t)$ such that each digraph D with $\delta^+(D) \geq f(s,t)$ admits a vertex partition (D_1, D_2) satisfying $\delta^+(D_1) \geq s$ and $\delta^+(D_2) \geq t$?

- Proposed at the Prague Midsummer Combinatorial Workshop in 1995.
- $f(1,1) = 3$ by Thomassen.
- ?$\exists$ $C (C = 10^5)$, s.t. $f(1,2) \leq C$ (Alon).
- Stiebitz proposed it again. (2016)

N. Alon, Splitting digraphs, Combinatorics, Probability and Computing 15 (2006), 933-937.

Given a digraph D and a bipartition (A, B) of $V(D)$. For a vertex $v \in V(D)$, denote by $d_A^+(v)$ the number of out-neighbors of v in A,

Theorem (Bai, W, Wu, Yang, EuJC, 2018)

For every $0 < \epsilon < \frac{1}{4}$, there exists an integer δ_0 such that every tournament T with $\delta^+(T) \geq \delta_0$ admits a bisection (A, B) with $\min\{d_A^+(v), d_B^+(v)\} \geq (\frac{1}{4} - \epsilon)d^+(v)$ for every $v \in V(T)$.

Corollary

Every tournament T with $\delta^+(T) \geq (4 + o(1))k$ admits a bisection (A, B) with $\min\{d_A^+(v), d_B^+(v)\} \geq k$ for every $v \in V(T)$.

- Recently, Alon, Bang-Jensen and Bessy (JGT, 2019) gave a better bound: $2k + c\sqrt{k}$. It is tight up to the value of c.

Theorem (Bai, W, Wu, Yang, EuJC, 2018)

For any positive integers $s \leq t$. If D is a bipartite tournament with $\delta^+(D) \geq t + \frac{(s+1)^4}{4s} - s$, then D has a bipartition (A, B) with $\delta^+(D[A]) \geq s$ and $\delta^+(D[B]) \geq t$.

Theorem (Bai, W, Wu, Yang, EuJC, 2018)

Every k-partite tournament D with $\delta^+(D) \geq t + ks(k-1)(s+1)^2$ has a bipartition (A, B) with $\delta^+(D[A]) \geq s$ and $\delta^+(D[B]) \geq t$.

Theorem (Bai, W, Wu, Yang, EuJC, 2018)

For every $0 < \epsilon < \frac{1}{4}$, there exists an integer δ_0 such that every digraph D with $\Delta^-(D) \leq \frac{e^{2\epsilon\delta^+(D)}}{16\delta^+(D)}$ admits a bisection (A, B) with $\min\{d_A^+(v), d_B^+(v)\} \geq (\frac{1}{4} - \epsilon)d_D^+(v)$ for every $v \in V(D)$.

- Weighted Local Lemma.
- Regular digraphs.

Problem (Fujita, Li, W, CPC, 2019)

Let s, t be integers with $s \geq t \geq 2$, and G be an edge-colored graph. Find $g(s,t)$ s.t. if $\delta^c(G) \geq g(s,t)$, then $V(G) = A \cup B$, $\delta^c(G[A]) \geq s$ and $\delta^c(G[B]) \geq t$.

Problem (Fujita, Li, W, CPC, 2019)

($g(s,t) = s + t + 1?$).

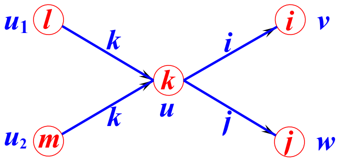
- Tight : Properly edge colored K_{s+t+1}.
- If it is true, then it implies Stiebitz's result.

Theorem (Fujita, Li, W, CPC, 2019)

$g(2,2) = 5$.

Proposition (Fujita, Li, W, CPC, 2019)

$f(1,2) \leq g(2,3)$.



Problem

For any natural number k, find a (smallest) number $f(k)$ such that every digraph D with $\delta^+(D) \geq f(k)$ contains k vertex-disjoint directed cycles.

Conjecture (Bermond, Thomassen, 1981)

$f(k) = 2k - 1$.

- Thomassen proved that such a function $f(k)$ exists.
- $f(k) \leq 64k$ (Alon).
- $f(k) \leq 18k$. (Bucic)
- True for $k \leq 3$ ($f(1) = 1, f(2) = 3, f(3) = 5$).

Proposition

If $g(s, 2) \leq s + 3$, then $f(k) \leq 3k - 1$.

Contents

Rainbow Hamiltonian cycles

Theorem (Alon, Pokrovskiy, Sudakov, 2017)

Every properly edge-colored complete graph K_n contains a rainbow cycle of length at least $n - o(n)$ when n is sufficient large.

Theorem (Balogh and Molla, 2019)

Every properly edge-colored complete graph K_n contains a rainbow cycle of length at least $n - C \log n \sqrt{n}$. when n is sufficient large.

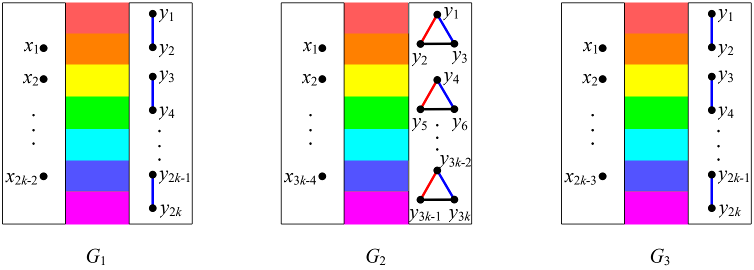
Theorem (Cheng, Sun, Tan, W, 2019)

Let G be a strongly edge-colored graph with minimum degree δ and order n, if $\delta > \frac{2n-1}{3}$, then G has a rainbow Hamiltonian cycle.

Strong edge coloring: every color class is an induced matching

Conjecture (Cheng, Sun, Tan, W, 2019)

Every strongly edge-colored graph G with minimum degree $\frac{n+1}{2}$ contains a rainbow Hamiltonian cycle.



Thank you for your attention!