Merging the A- and Q-spectral theories for digraphs

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Outline

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- $G_1 \vee G_2$: it is the digraph such that $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{(u, v), (v, u) : u \in V(G_1),\ v \in V(G_2)\}$.

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- Complete digraph $\overleftrightarrow{K_n}$: for two arbitrary distinct vertices $v_i, v_j \in V(\overleftrightarrow{K_n})$, there are arcs (v_i, v_j) and $(v_j, v_i) \in E(\overleftrightarrow{K_n})$.

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- r-regular digraph: if all vertices have outdegree r and indegree r.
- Adjacency matrix: $A(G) = (a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & (v_i, v_j) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

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- Out-degree diagonal matrix: $D(G) = \mathrm{diag}(d_1^+, d_2^+, \ldots, d_n^+)$.

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- Note that $A(G) = A_0(G)$, $D(G) = A_1(G)$ and $Q(G) = 2A_{\frac{1}{2}}(G)$. Since $A_{\frac{1}{2}}(G)$ is essentially equivalent to $Q(G)$, in this paper we take $A_{\frac{1}{2}}(G)$ as an exact substitute for $Q(G)$. With this caveat, one sees that $A_\alpha(G)$ seamlessly joins $A(G)$ with $Q(G)$, and we may study the adjacency spectral properties and signless Laplacian spectral properties of a digraph in a unified way.

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- A_α spectral radius: The largest modulus of the eigenvalues of $A_\alpha(G)$, denoted by $\lambda_\alpha(G)$.
- The Perron vector of $A_\alpha(G)$: the positive unit eigenvector corresponding to $\lambda_\alpha(G)$ (when G is strongly connected.)

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Problem

One of the central issues in extremal spectral graph theory is:
for a graph matrix, determine the maximum or minimum spectral radius over various families of graphs.

A_α spectral radius of graphs

- In 2017, Nikiforov [1] proposed to study the convex combinations $A_\alpha(H) = \alpha D(H) + (1 - \alpha)A(H)$ of an undirected graph H.

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  1) $\rho_\alpha(T) \leq \rho_\alpha(K_{1,n-1})$ with equality if and only if $G \cong K_{1,n-1}$.

  2) $\rho_\alpha(G) \geq \rho_\alpha(P_n)$ with equality if and only if $G \cong P_n$.

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V. Nikiforov, G. Pastén, O. Rojo, R.L. Soto, On the A_α-spectra of trees, Linear Algebra Appl., 520 (2017), 286-305.

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In 2018, Nikiforov and Rojo [1] determined the unique graph with maximum A_α spectral radius among all connected graphs of order n and diameter at least k.

- $\rho_\alpha(G) \le \rho_\alpha(B_{n-k+2,\lfloor k/2\rfloor,\lceil k/2\rceil})$ with equality if and only if $G \cong B_{n-k+2,\lfloor k/2\rfloor,\lceil k/2\rceil})$,

  where $B_{p,q,r}$ denotes the graph obtained from a complete graph K_p by deleting an edge and attaching paths P_q and P_r to its ends.

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In 2018, Xue et al. [4] determined the unique graph with maximum A_α spectral radius among all connected graphs with diameter d and determined the unique graph with minimum A_α spectral radius among all connected graphs with given clique number r.

- If G is a connected graph with diameter $d \geq 2$, then $\rho_\alpha(G) \leq \rho_\alpha(K_{n-d}(\lfloor d/2 \rfloor, \lceil d/2 \rceil))$ with equality if and only if $G \cong K_{n-d}(\lfloor d/2 \rfloor, \lceil d/2 \rceil))$,

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- If $G \in C_n^r$, then $\rho_\alpha(G) \geq \rho_\alpha(K(n-r))$ with equality if and only if $G \cong K(n-r)$,

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Adjacency spectral radius of digraphs

Theorem 1 (B. Mohar, LAA, 2010)

Let $\varrho(G)$ be the spectral radius of a simple digraph G, $\chi(G)$ be the dichromatic number of G. Then $\varrho(G) \geq \chi(D) - 1$.
If G is strongly connected, then the equality holds if and only if G is one of the digraphs listed in following:
(a) $\chi(G) = 2$ and G is a directed cycle of length $n \geq 2$.
(b) $\chi(G) = 3$ and G is a bidirected cycle of odd length $n \geq 3$.
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B. Mohar, Eigenvalues and colorings of digraphs, Linear Algebra Appl., 432 (2010,) 2273-2277.

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Theorem 2 (E. Gudiño, J. Rada, LAA, 2010)

Let G be a simple digraph with n vertices and c_2 closed walks of length 2. Then $\varrho(G) \geq \frac{c_2}{n}$.

Equality holds if and only if $G = \overleftrightarrow{D} + \{$possibly some arcs that do not belong to cycle$\}$, where D is a $\frac{c_2}{n}$-regular graph.

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E. Gudiño, J. Rada, A lower bound for the spectral radius of a digraph, Linear Algebra Appl., 433 (2010), 233 – 240.

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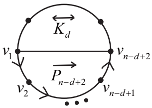
Theorem 3 (H.Q. Lin, J.L. Shu, et al., DM, 2012)

The digraph $B_{n,d}$ is the unique digraph which has the minimum spectral radius among all strongly connected digraphs with the clique number $d \geq 2$, where $B_{n,d}$ is a digraph obtained by adding a directed path $\overrightarrow{P}_{n-d+2} = v_1 v_2 \dots v_{n-d+2}$ to a clique $\overleftrightarrow{K_d}$ such that $V(\overleftrightarrow{K_d}) \cap V(\overrightarrow{P}_{n-d+2}) = \{v_{n-d+2}, v_1\}$, $V(B_{n,d}) = \{v_1, v_2, \dots, v_n\}$.



Adjacency spectral radius of digraphs

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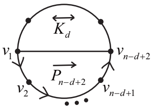

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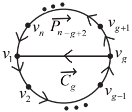

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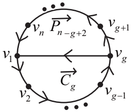

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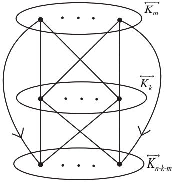

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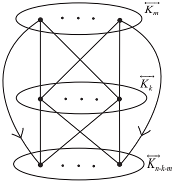

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Signless Laplacian spectral radius of digraphs

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Let $G = (V, E)$ be a strongly connected digraph on n vertices, $q(G)$ be the signless Laplacian spectral radius of G. Then

$$\min\{d_i^+ + m_i^+ : v_i \in V\} \le q(G) \le \max\{d_i^+ + m_i^+ : v_i \in V\}.$$

$$\min\{d_i^+ + d_j^+ : (v_i, v_j) \in E\} \le q(G) \le \max\{d_i^+ + d_j^+ : (v_i, v_j) \in E\}.$$

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Moreover, if $i = 1$, the equality holds if and only if G is a regular digraph. If $2 \leq i \leq n$, the equality holds if and only if G is a regular digraph or a bidegreed digraph in which $d_1^+ = d_2^+ = \ldots = d_{i-1}^+ = n - 1$ and $d_i^+ = d_2^+ = \ldots = d_n^+ = \delta^+$.

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A_α spectral radius of digraphs

To understand to what extent each of the summands $A(G)$ and $D(G)$ determines the properties of $Q(G)$, Liu et al. [1] defined the matrix $A_\alpha(G)$ as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \quad 0 \le \alpha \le 1.$$

Many facts suggest that the study of the family $A_\alpha(G)$ is long due.

Theorem 15 (J.P. Liu, X.Z. Wu, et al., LAA, 2019)

Among all digraphs in with given dichromatic number $k \ge 2$, the digraph $\mathcal{T}^{*k}_n$ is the unique digraph which has the maximal A_α spectral radius.

Let $[V_1, V_2]$ denote the arcs between V_1 and V_2. Let $\mathcal{T}_n^k$ denote the set of digraphs with $V(\mathcal{T}_n^k) = V^1 \cup V^2 \cup \cdots V^k$, where V^i $(i = 1, 2, \ldots, k)$ is a transitive tournament and $[V^i, V^j] = \{(v_s^i, v_t^j), (v_t^j, v_s^i) : v_s^i \in V^i, v_t^j \in V^j\}$, and $\mathcal{T}^{*k}_n$ denotes the digraph in $\mathcal{T}_n^k$ with $||V^i| - |V^j|| \le 1$ $(i, j \in \{1, 2, \ldots, k\})$.

A_α spectral radius of digraphs

To understand to what extent each of the summands $A(G)$ and $D(G)$ determines the properties of $Q(G)$, Liu et al. [1] defined the matrix $A_\alpha(G)$ as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \quad 0 \le \alpha \le 1.$$

Many facts suggest that the study of the family $A_\alpha(G)$ is long due.

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Let $[V_1, V_2]$ denote the arcs between V_1 and V_2. Let $\mathcal{T}^k_n$ denote the set of digraphs with $V(\mathcal{T}^k_n) = V^1 \cup V^2 \cup \cdots V^k$, where V^i $(i = 1, 2, \ldots, k)$ is a transitive tournament and $[V^i, V^j] = \{(v^i_s, v^j_t), (v^j_t, v^i_s) : v^i_s \in V^i, v^j_t \in V^j\}$, and $\mathcal{T}^{*k}_n$ denotes the digraph in $\mathcal{T}^k_n$ with $\|V^i| - |V^j\| \le 1$ $(i, j \in \{1, 2, \ldots, k\})$.

J.P. Liu, X.Z. Wu, J.S. Chen, B.L. Liu, The A_α spectral radius characterization of some digraphs, Linear Algebra Appl., 563 (2019), 63-74.

Outline

1. Basic notation and Terminology
2. Background and known results
   - A_α Spectral radius of graphs
   - Adjacency spectral of digraphs
   - Signless Laplacian spectral of digraphs
3. Our Main Results in this paper

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- Finally, we characterize the extremal digraphs which achieve the maximum A_α spectral radius among all strongly connected digraphs with given vertex connectivity or arc connectivity.

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W.G. Xi, W. So, L.G. Wang, Merging the A- and Q-spectral theories for digraphs, arXiv:1810.11669.

Some lemmas

Lemma 16 (R.A. Horn, C.R. Johnson, Matrix Analysis, 1985)

Let $M = (m_{ij})$ be an $n \times n$ nonnegative matrix, $R_i(M)$ be the i-th row sum of M. Then

$$\min\{R_i(M) : 1 \leq i \leq n\} \leq \rho(M) \leq \max\{R_i(M) : 1 \leq i \leq n\}.$$

Moreover, if M is irreducible, then either one equality holds if and only if $R_1(M) = R_2(M) = \ldots = R_n(M)$.

Some lemmas

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Lemma 17 (R.A. Horn, C.R. Johnson, Matrix Analysis, 1985)

Let A and B be a nonnegative matrix. If $0 \leq A \leq B$, then $\rho(A) \leq \rho(B)$. Furthermore, if B is irreducible and $0 \leq A < B$, then $\rho(A) < \rho(B)$.

By Lemma 17, we have the following results in terms of A_α spectral radius of digraphs.

Corollary 18

Let G be a digraph and H be a spanning subdigraph of G. Then
(i) $\lambda_\alpha(G) \geq \lambda_\alpha(H)$.
(ii) If G is strongly connected, and H is a proper subdigraph of G, then $\lambda_\alpha(G) > \lambda_\alpha(H)$.

From Lemma 16 and Corollary 18, we can easily get the following corollary.

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From Lemma 16 and Corollary 18, we can easily get the following corollary.

Corollary 19

Let G be a strongly connected digraph. Then $1 \leq \lambda_\alpha(G) \leq n-1$, $\lambda_\alpha(G) = n-1$ *if and only if* $G \cong \overleftrightarrow{K_n}$, *and* $\lambda_\alpha(G) = 1$ *if and only if* $G \cong \overrightarrow{C_n}$.

Lemma 20 (R.A. Horn, C.R. Johnson, Matrix Analysis, 1985)

Let B be a nonnegative matrix and $X = (x_1, x_2, \ldots, x_n)^T$ be any nonzero nonnegative vector. If $\beta \geq 0$ such that $BX \geq \beta X$, then $\rho(B) \geq \beta$. Furthermore, if B is irreducible and $BX > \beta X$, then $\rho(B) > \beta$.

By Lemma 20, we have the following results in terms of A_α spectral radius of digraphs.

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By Lemma 20, we have the following results in terms of A_α spectral radius of digraphs.

Corollary 21

Let G be a strongly connected digraph. Then $\lambda_\alpha(G) > \alpha\Delta^+$.

Furthermore, we can prove the following result on A_α spectral radius of digraphs

Lemma 22 (W.G. Xi, W. So, L.G. Wang, 2018)

Let $G = (V(G), E(G))$ be a strongly connected digraph on n vertices, v_p, v_q be two distinct vertices of $V(G)$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $A_\alpha(G)$. Suppose that $v_1, v_2, \ldots, v_t \in N^-_{v_p} \setminus \{N^-_{v_q} \cup \{v_q\}\}$, where $1 \le t \le d^-_p$. Let $H = G - \{(v_i, v_p) : i = 1, 2 \ldots, t\} + \{(v_i, v_q) : i = 1, 2 \ldots, t\}$. If $x_{v_q} \ge x_{v_p}$, then $\lambda_\alpha(H) \ge \lambda_\alpha(G)$. Furthermore, if H is strongly connected and $x_{v_q} > x_{v_p}$, then $\lambda_\alpha(H) > \lambda_\alpha(G)$.

Lemma 23 (W.G. Xi,W. So, L.G. Wang, 2018)

Let G $(\neq \overrightarrow{C_n})$ *be a strongly connected digraph with* $V(G) = \{v_1, v_2, \cdots, v_n\}$, $(v_i, v_j) \in E(G)$ *and* $w \notin V(G)$, $G^w = (V(G^w), E(G^w))$ *with* $V(G^w) = V(G) \cup \{w\}$, $E(G^w) = E(G) - \{(v_i, v_j)\} + \{(v_i, w), (w, v_j)\}$. *Then* $\lambda_\alpha(G) \geq \lambda_\alpha(G^w)$.

Lemma 23 (W.G. Xi,W. So, L.G. Wang, 2018)

Let G $(\neq \overrightarrow{C_n})$ be a strongly connected digraph with $V(G) = \{v_1, v_2, \cdots, v_n\}$, $(v_i, v_j) \in E(G)$ and $w \notin V(G)$, $G^w = (V(G^w), E(G^w))$ with $V(G^w) = V(G) \cup \{w\}$, $E(G^w) = E(G) - \{(v_i, v_j)\} + \{(v_i, w), (w, v_j)\}$. Then $\lambda_\alpha(G) \geq \lambda_\alpha(G^w)$.

Lemma 24 (W.G. Xi,W. So, L.G. Wang, 2018)

Let G be a strongly connected digraph with order n and arc connectivity $k \geq 1$, and S be an arc cut set of G of size k such that $G - S$ has exactly two strongly connected components, say G_1 and G_2 with $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$, where $n_1 + n_2 = n$. If $d_v^+ > k$ and $d_v^- > k$ for each vertex $v \in V(G)$, then $n_1 \geq k + 2$, $n_2 \geq k + 2$.

Lemma 25 (Bondy, Murty, Graph Theory with Applications)

Let G be a strongly connected digraph with $\kappa(G) = d$. Suppose that S is a d-vertex cut of G and $G_1, G_2, \ldots, G_t$ are the strongly connected components of $G - S$. Then there exists an ordering of $G_1, G_2, \ldots, G_t$ such that for $1 \leq i \leq t$ and any $v \in V(G_i)$, every tail of v is in $\bigcup_{j=1}^{i} G_j$.

- $\mathcal{G}_{n,g}$: the set of strongly connected digraphs on n vertices with girth $g \geq 2$. If $g = n$, then $\mathcal{G}_{n,g} = \{\overrightarrow{C_n}\}$ and $\lambda_\alpha(\overrightarrow{C_n}) = 1$. Thus we only need to consider the cases $2 \leq g \leq n-1$.

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- $\mathcal{C}_{n,d}$: the set of strongly connected digraphs on n vertices with clique number d. If $d = n$, then $\mathcal{C}_{n,d} = \{\overleftrightarrow{K_n}\}$ and $\lambda_\alpha(\overleftrightarrow{K_n}) = n-1$. If $d = 1$, then $\overrightarrow{C_n} \in \mathcal{C}_{n,d}$ and $\lambda_\alpha(\overrightarrow{C_n}) = 1$. By Corollary 19, for any $G \in \mathcal{C}_{n,d}$, $\lambda_\alpha(G) \geq 1 = \lambda_\alpha(\overrightarrow{C_n})$ with equality if and only if $G \cong \overrightarrow{C_n}$. Thus we only need to consider the cases $2 \leq d \leq n-1$.

- $\mathcal{D}_{n,k}$: the set of strongly connected digraphs with order n and vertex connectivity $\kappa(G) = k \geq 1$. If $k = n - 1$, then $\mathcal{D}_{n,k} = \{\overleftrightarrow{K_n}\}$. So we only consider the cases $1 \leq k \leq n - 2$.

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- For $1 \leq m \leq n-k-1$, $K(n,k,m)$: the digraph $\overleftrightarrow{K_k} \vee (\overleftrightarrow{K}_{n-k-m} \cup \overleftrightarrow{K_m}) + E$, where $E = \{(u,v) | u \in V(\overleftrightarrow{K_m}), v \in V(\overleftrightarrow{K}_{n-k-m})\}$.

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- For $1 \leq m \leq n - k - 1$, $K(n, k, m)$: the digraph $\overleftrightarrow{K_k} \vee (\overleftrightarrow{K}_{n-k-m} \cup \overleftrightarrow{K_m}) + E$, where $E = \{(u, v) | u \in V(\overleftrightarrow{K_m}), v \in V(\overleftrightarrow{K}_{n-k-m})\}$.
- $\mathcal{L}_{n,k}$ the set of strongly connected digraphs with order n and arc connectivity $\kappa'(G) = k \geq 1$. If $\kappa'(G) = k = n - 1$, then $\mathcal{L}_{n,k} = \{\overleftrightarrow{K_n}\}$. So we only consider the cases $1 \leq k \leq n - 2$.

Our Main Results in this paper

Theorem 26 (W.G. Xi, W. So, L.G. Wang, 2018)

Let $2 \leq g \leq n-1$ *and* $G \in \mathcal{G}_{n,g}$. *Then* $\lambda_\alpha(G) \geq \lambda_\alpha(C_{n,g}) > 1$, *with equality if and only if* $G \cong C_{n,g}$.

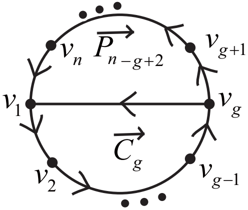

Figure 3.1: $C_{n,g}$

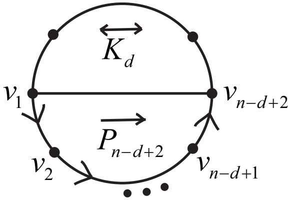

Figure 3.2: $B_{n,d}$

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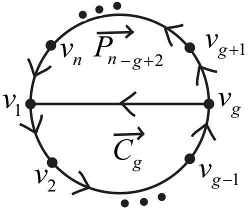
Let $2 \le g \le n-1$ and $G \in \mathcal{G}_{n,g}$. Then $\lambda_\alpha(G) \ge \lambda_\alpha(C_{n,g}) > 1$, with equality if and only if $G \cong C_{n,g}$.

Theorem 27 (W.G. Xi, W. So, L.G. Wang, 2018)

Let $2 \le d \le n-1$ and $G \in C_{n,d}$. Then $\lambda_\alpha(G) \ge \lambda_\alpha(B_{n,d})$, with equality if and only if $G \cong B_{n,d}$.



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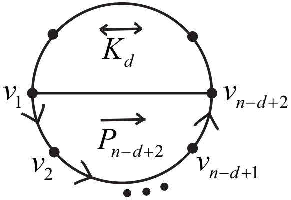

Figure 3.2: $B_{n,d}$

Theorem 28 (W.G. Xi, W. So, L.G. Wang, 2018)

Let n, k, m be positive integers such that $1 \le k \le n-2$ and $1 \le m \le n-k-1$. Then

$$\lambda_\alpha(K(n,k,m)) = \frac{n-2-\alpha m+\alpha n+\sqrt{(1-\alpha)^2n^2+(6\alpha-2\alpha^2-4)mn+(2-\alpha)^2m^2+4(1-\alpha)km}}{2}.$$

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Theorem 29 (W.G. Xi, W. So, L.G. Wang, 2018)

Let G be a strongly connected digraph. If $G \ncong \overleftrightarrow{K_n}$, and $G \ncong K(n, n-2, 1)$. Then

$$\lambda_\alpha(G) > \frac{n+\alpha n-2-\alpha+\sqrt{(1-\alpha)^2n^2+2\alpha(1-\alpha)n+\alpha^2+4\alpha-4}}{2}.$$

Theorem 30 (W.G. Xi, W. So, L.G. Wang, 2018)

Let n, k be positive integers such that $1 \leq k \leq n-2$, $G \in \mathcal{D}_{n,k}$. Then
(i) For $\alpha = 0$, $\lambda_\alpha(G) \leq \frac{n-2+\sqrt{n^2-4n+4k+4}}{2}$, with equality if and only if $G \cong K(n, k, n-k-1)$ or $G \cong K(n, k, 1)$.
(ii) For $0 < \alpha < 1$,
$\lambda_\alpha(G) \leq \frac{n-2+\alpha+\alpha k+\sqrt{n^2+(2\alpha-4-2\alpha k)n+\alpha^2+\alpha^2k^2-4\alpha+2\alpha^2k-4\alpha k+4k+4}}{2}$, with equality if and only if $G \cong K(n, k, n-k-1)$.

Theorem 31 (W.G. Xi, W. So, L.G. Wang, 2018)

Let $G \in \mathcal{L}_{n,k}$. Then
(i) For $\alpha = 0$, $\lambda_\alpha(G) \le \lambda_\alpha(K(n,k,1)) = \lambda_\alpha(K(n,k,n-k-1))$, with equality if and only if $G \cong K(n,k,n-k-1)$ or $G \cong K(n,k,1)$.
(ii) For $0 < \alpha < 1$, $\lambda_\alpha(G) \le \lambda_\alpha(K(n,k,n-k-1))$, with equality if and only if $G \cong K(n,k,n-k-1)$.

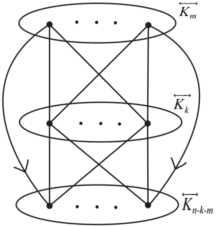

Figure 3.3: The digraph $K(n,k,m)$

Thank You!!!