Log Behavior of Partition Function

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Outline

Introduction

Recall that a positive function f is called log-convex on a real interval $I = [a, b]$, if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq f(x)^{\lambda} f(y)^{1-\lambda}, \tag{1}$$

It is known that a positive function f is log-convex if and only if $(\log f(x))'' \geq 0$.

Introduction

Definition

A sequence $\{a_i\}_{0\le i\le m}$ of real numbers is said to be log-convex if

$$a_i^2 \le a_{i+1}a_{i-1}$$

for all $1 \le i \le m-1$, and is said to strictly log-convex if

$$a_i^2 < a_{i+1}a_{i-1}.$$

Introduction

Definition

A sequence $\{a_i\}_{0\le i\le m}$ *is called unimodal if there exists* k *such that*

$$a_0 \le \cdots \le a_k \ge \cdots \ge a_m,$$

and is called strictly unimodal if

$$a_0 < \cdots < a_k > \cdots > a_m.$$

Introduction

Definition

A sequence $\{a_i\}_{0\le i\le m}$ of real numbers is said to be log-concave if

$$a_i^2 \geq a_{i+1}a_{i-1}$$

for all $1 \le i \le m-1$, and is said to be strictly log-concave if

$$a_i^2 > a_{i+1}a_{i-1}.$$

Example: The sequence

$$1, 3, 5, 9, 5, 3, 1$$

is symmetric unimodal but not log-concave.

Completely Monotonic Functions

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{2}$$

for $x \in I$ and all integers $n \geq 0$.

A positive function f is said to be logarithmically completely monotonic on an interval I if $\log f$ satisfies

$$(-1)^n [\log f(x)]^{(n)} \geq 0 \tag{3}$$

for $x \in I$ and all the integers $n \geq 1$. A logarithmically completely monotonic function is completely monotonic, but the converse is not necessarily the case.

Infinite Log-monotonicity

In 2014, Chen, Guo and Wang defined that a sequence $\{a_n\}_{n\geq 0}$ is log-monotonic of order k if for any $r \leq k+1$,

$$(-1)^r \Delta^r \log a_n > 0.$$

A sequence $\{a_n\}_{n\geq 0}$ is called infinitely log-monotonic if it is log-monotonic of order k for all integers $k \geq 1$.

Infinite Log-monotonicity

We have the following relation between the completely monotonic functions and infinitely log-monotonic sequences.

Theorem (Chen, Guo and Wang, Adv. Appl. Math., 2014)

Assume that $f(x)$ is a function such that $[\log f(x)]''$ is completely monotonic for $x \geq 1$ and $a_n = f(n)$ for $n \geq 1$. Then the sequence $\{a_n\}_{n\geq 1}$ is infinitely log-monotonic.

Definition

Log-monotonic sequences of order two are of special interest. A sequence $\{a_n\}_{n\geq 0}$ is log-monotonic of order two if and only if it is log-convex and the ratio sequence $\{a_{n+1}/a_n\}_{n\geq 0}$ is log-concave. A sequence $\{a_n\}_{n\geq 0}$ is said to be ratio log-concave if $\{a_{n+1}/a_n\}_{n\geq 0}$ is log-concave. Similarly, a sequence $\{a_n\}_{n\geq 0}$ is called ratio log-convex if the ratio sequence $\{a_{n+1}/a_n\}_{n\geq 0}$ is log-convex.

Ratio Log-Concavity implies the Conjectures of Sun

Theorem (Chen, Guo and Wang, Adv. Appl. Math., 2014)

Assume that k is a positive integer. If a sequence $\{a_n\}_{n\geq k}$ is ratio log-concave and

$$\frac{\sqrt[k+1]{a_{k+1}}}{\sqrt[k]{a_k}} > \frac{\sqrt[k+2]{a_{k+2}}}{\sqrt[k+1]{a_{k+1}}}, \tag{4}$$

then the sequence $\{\sqrt[n]{a_n}\}_{n\geq k}$ is strictly log-concave.

Theorem (Chen, Guo and Wang, Adv. Appl. Math., 2014)

Assume that k is a positive integer. If a sequence $\{a_n\}_{n\geq k}$ is ratio log-convex and

$$\frac{\sqrt[k+1]{a_{k+1}}}{\sqrt[k]{a_k}} < \frac{\sqrt[k+2]{a_{k+2}}}{\sqrt[k+1]{a_{k+1}}},$$

then the sequence $\{\sqrt[n]{a_n}\}_{n\geq k}$ is strictly log-convex.

Introduction

(1) Finite difference of the logarithm of partition function

(2) Higher order Turán inequality and Riemann hypothesis

(3) Log-behavior of overpartition function

(4) Higher order Turán inequality for combinatorial sequences

Outline

Parition

A partition of a positive integer n is a nonincreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = n.$$

Let $p(n)$ denote the number of partitions of n.

Conjecture

In 2010, Chen proposed the following conjecture

Conjecture (Chen, 2010)

For $n > 25$,

$$\frac{p(n-1)}{p(n)} < \frac{p(n)}{p(n+1)} \tag{5}$$

and for $n \geq 2$,

$$\frac{p(n)}{p(n+1)} < \frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right). \tag{6}$$

In 2015, DeSalvo and Pak proved these two conjectures.

DeSalvo and Pak's proof is based on Hardy-Ramanujan-Rademacher formula for $p(n)$ and Lehmer's error bound. Recall that Hardy-Ramanujan-Rademacher formula for $p(n)$ states that for $n \geq 1$,

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k} \right] + R_2(n, N),$$

where $A_k(n)$ is an arithmetic function, $R_2(n, N)$ is the remainder term and

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}. \tag{7}$$

Lehmer gave an error bound for $R_2(n, N)$.

$$|R_2(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu(n)}\right)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)}\right)^2 \right],$$

which is valid for all positive integers n and N.

Theorem (Bessenrodt and Ono, Ann. Combin., 2016)

For any integers a and b satisfying $a,\ b > 1$ and $a + b > 9$, we have

$$p(a)p(b) > p(a+b).$$

Conjecture (Desalvo and Pak, Ramanujan J., 2015)

For $n \geq 45$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}.$$

This conjecture has been proved by Chen, Wang and Xie.

Notice that the above results can be rewritten as

$$0 \leq -\Delta^2 \log p(n) < \log\left(1 + \frac{\pi}{\sqrt{24} n^{3/2}}\right).$$

What happens with $\Delta^n \log p(n)$?

Motivation

(1) In 1977, Good conjectured that $\Delta^r p(n)$ alternates in sign up to a certain value $n = n(r)$, and then it stays positive;

(2) In 1978, Using the Hardy-Rademacher series for $p(n)$, Gupta proved that for any given r, $\Delta^r p(n) > 0$ for sufficiently large n.

(3) In 1988, Odlyzko proved the conjecture of Good and obtained the following asymptotic formula for $n(r)$:

$$n(r) \sim \frac{6}{\pi^2} r^2 \log^2 r \quad \text{as } r \to \infty.$$

(4) In 1991, Using the WKB method on the difference equation for $\Delta^r p(n)$, Knessl and Keller obtained an approximation $n(r)'$ for $n(r)$ for which $|n(r)' - n(r)| \leq 2$ up to $r = 75$.

Finite differencep of $\log p(n)$

Theorem (Chen, Wang and Xie, Math. Comp., 2016)

For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,

$$0 < (-1)^{r-1}\Delta^r \log p(n) < \log\left(1 + \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+1)^{r-\frac{1}{2}}}\right).$$

Question:

Find a sharper lower bound for $(-1)^{r-1}\Delta^r \log p(n)$.

Outline

Introduction

The Turán inequalities and the higher order Turán inequalities arise in the study of Maclaurin coefficients of an entire function in the Laguerre-Pólya class. A real sequence $\{a_n\}$ is said to satisfy the Turán inequalities if for $n \geq 1$,

$$a_n^2 - a_{n-1}a_{n+1} \geq 0.$$

It is said to satisfy the higher order Turán inequalities if for $n \geq 1$,

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_na_{n+2}) - (a_na_{n+1} - a_{n-1}a_{n+2})^2 \geq 0.$$

Background

A real entire function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \qquad (8)$$

is said to be in the Laguerre-Pólya class, denoted $\psi(x) \in \mathcal{LP}$, if it can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k},$$

where c, β, x_k are real numbers, $\alpha \geq 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. These functions are the only ones which are uniform limits of polynomials whose zeros are real.

Background

The Jensen polynomial of degree d and shift n of an arbitrary sequence $\{\alpha(0), \alpha(1), \alpha(2), \ldots\}$ of real numbers is the polynomial

$$J_{\alpha}^{d,n}(x) = \sum_{k=0}^{d} \binom{d}{k} \alpha(n+k) x^k$$

In 1913, Jensen proved that a real entire function $\psi(x)$ belongs to $\mathcal{LP}$ class if and only if for any positive integer n, the n-th associated Jensen polynomial with γ_k

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \tag{9}$$

has only real zeros.

Background

Pólya and Schur showed that if a real entire function $\psi(x)$ belongs to the $\mathcal{LP}$ class, then its Maclaurin coefficient sequence satisfies the Turán inequalities

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0 \tag{10}$$

for $k \geq 1$. Moreover, if a real entire function $\psi(x)$ belongs to the $\mathcal{LP}$ class, then the n-th Jensen polynomial associated with $\psi^{(m)}$

$$g_{n,m}(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_{k+m} x^k, \tag{11}$$

has only real zeros for any nonnegative integers n and m.

Background

In 1998, Dimitrov observed that for a real entire function $\psi(x)$ in the $\mathcal{LP}$ class, the Maclaurin coefficients satisfy the higher order Turán inequalities

$$4(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}) - (\gamma_k\gamma_{k+1} - \gamma_{k-1}\gamma_{k+2})^2 \geq 0 \quad (12)$$

for $k \geq 1$. This fact follows from a theorem of Mařík stating that if a real polynomial

$$\sum_{k=0}^{n} \binom{n}{k} a_k x^k \quad (13)$$

of degree $n \geq 3$ has only real zeros, then $a_0, a_1, \ldots, a_n$ satisfy the higher order Turán inequalities.

Background

The $\mathcal{LP}$ class is closely related to the Riemann hypothesis. Let ζ denote the Riemann zeta-fucntion and Γ be the gamma-function. Recall that the Riemman ξ-function is defined by

$$\xi(iz) = \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-z/2-1/4}\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)\zeta\left(z+\frac{1}{2}\right). \qquad (14)$$

It is well known that the Riemann hypothesis holds if and only if the Riemann ξ-function belongs to the $\mathcal{LP}$ class.

Background

If the Riemann hypothesis is true, then the Maclaurin coefficients of the Riemann ξ-function satisfy both the Turán inequalities and the higher order Turán inequalities.

- In 1986, Csordas, Norfolk and Varga proved that the coefficients of the Riemann ξ-function indeed satisfy the Turán inequalities, confirming a conjecture of Pólya.
- In 2011, Dimitrov and Lucas showed that the coefficients of the Riemann ξ-function satisfy the higher order Turán inequalities without the Riemann hypothesis.

These results provided support to the positivity of Riemann hypothesis.

Main Result

Theorem (Chen, Jia and Wang, Trans. Amer. Math. Soc., 2019)

Let $p(n)$ be the partition function. Then, for $n \geq 95$,

$$\begin{aligned} 4(p(n)^2 - p(n-1)p(n+1))(p(n+1)^2 - p(n)p(n+2)) \\ -(p(n)p(n+1) - p(n-1)p(n+2))^2 > 0. \end{aligned} \tag{15}$$

Sketch of Proof

Because we have to deal with the product and the minus of $p(n)$ simultaneously, it is no possible to directly apply the method in the calculation of the finite difference of $\log p(n)$. We translate it into the following form: for $n \geq 95$,

$$4(1-a_n)(1-a_{n+1}) - (1-a_n a_{n+1})^2 > 0,$$

where $a_n = \frac{p(n+1)p(n-1)}{p(n)^2}$.

Sketch of Proof

Lemma

If $0 < \alpha < \beta < Q(\alpha) < 1$, *where* $Q(\alpha) = \frac{3\alpha+2\sqrt{-(\alpha-1)^3}-2}{\alpha^2}$, *then*

$$4(1-\alpha)(1-\beta) - (1-\alpha\beta)^2 > 0.$$

Chen, Wang and Xie have proved that $a_n < a_{n+1}$ for $n \geq 45$. Hence, the remaining is to prove that for $n \geq 95$,

$$a_{n+1} < \frac{3a_n + 2\sqrt{-(a_n-1)^3} - 2}{a_n^2}.$$

Real Zero

Theorem (Chen, Jia and Wang, Trans. Amer. Math. Soc., 2019)

For $n \geq 95$, the polynomial

$$p(n-1) + 3p(n)x + 3p(n+1)x^2 + p(n+2)x^3,$$

has only real zeros.

Since the positivity of the discriminant of the above polynomials is equivalent to the higher order Turán inequalities of $p(n)$.

Conjecture

Conjecture (Chen-Jia-Wang Conjecture)

For any positive integer $m \geq 4$, there exists a positive integer $N(m)$ such that for any $n \geq N(m)$, the polynomial

$$\sum_{k=0}^{m} \binom{m}{k} p(n+k) x^k$$

has only real zeros.

Griffin, Ono, Rolen and Zagier proved this conjecture for sufficiently large n.

Approach to RH

Riemann hypothesis is equivalent to the hyperbolicity of the polynomials $g_{n,m}(x)$ for all nonnegative integers m and n.

Theorem (Griffin, Ono, Rolen and Zagier, PNAS, 2019)

If $n \geq 1$, then $g_{n,m}(x)$ is hyperbolic for all sufficiently large m.

Theorem (Griffin, Ono, Rolen and Zagier, PNAS, 2019)

For $1 \leq n \leq 8$, $g_{n,m}(x)$ is hyperbolic for all m.

Approach to RH

Theorem (Griffin, Ono, Rolen and Zagier, PNAS, 2019)

Let $\alpha(n)$, $A(n)$, and $\delta(n)$ be three sequences of positive real numbers with $\delta(n)$ tending to zero and satisfying

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = A(n)j - \delta(n)^2 j^2 + o(\delta(n)^d) \quad \text{as} \quad n \to \infty$$

for some integer $d \geq 1$ and all $0 \leq j \leq d$. Then, we have

$$\lim_{n\to\infty}\left(\frac{\delta(n)^{-d}}{\alpha(n)} J_{\alpha}^{d,n}\left(\frac{\delta(n)x-1}{\exp(A(n))}\right)\right) = H_d(x)$$

uniformly for x in any compact subset of $\mathbb{R}$, where $H_d(x)$ is the Hermite polynomials

Approach to RH

Since the Hermite polynomials have distinct roots, and since this property of a polynomial with real coefficients is invariant under small deformation, one can immediately deduce the following result.

Theorem (Griffin, Ono, Rolen and Zagier, PNAS, 2019)

The Jensen polynomials $J_{\alpha}^{d,n}(x)$ for a sequence α satisfying the conditions in the above theorem are hyperbolic for all but finitely many values n.

The method of proof is rooted in the newly discovered phenomenon that these polynomials are nicely approximated by Hermite polynomials. Furthermore, it is shown that this method applies to a large class of related problems.

Approach to RH

The partition functions $p(n)$ are the Fourier coefficients of a modular form, namely,

$$\frac{1}{\eta(\tau)} = \sum_{n=0}^{\infty} p(n) q^{n-\frac{1}{24}}.$$

where $\eta(\tau) = q^{1/24} \prod (1 - q^n)$ is the Dedekind eta-function.

Theorem (Griffin, Ono, Rolen and Zagier, PNAS, 2019)

Assume that $f(\tau)$ is a modular form of weight k on $SL_2(Z)$ and with a pole of order $m > 0$ at infinity. If

$$f(\tau) = \sum_{n \in -m + \mathbb{Z}_{\geq 0}} a_f(n) q^n \quad (m \in \mathbb{Q}_{\geq 0}, a_f(-m) \neq 0),$$

then for any fixed $d \geq 1$, the Jensen polynomials $J_{a_f(n)}^{d,n}(x)$ are hyperbolic for all sufficiently large n.

Approach to RH

For partition function $p(n)$, by the previous results, one can choose

$$A(n) = \frac{2\pi}{\sqrt{24n-1}} - \frac{24}{24n-1}$$

and

$$\delta(n) = \sqrt{\frac{12\pi}{(24n-1)^{3/2}} - \frac{288}{(24n-1)^2}}.$$

We can observe that the degree 2 and 3 partition Jensen polynomials are modeled by $H_2(X) = X^2 - 2$ and $H_3(X) = X^3 - 6X$.

Approach to RH

For the threshold of partition function, Larson and Wagner gave the exactly values of $N(4) = 206$ and $N(5) = 381$ and

$$N(m) = (3m)^{24m}(50m)^{3m^2}.$$

They also prove the following theorem.

Theorem (Larson and Wagner, Res. Number Th., 2019)

Let $u_n = p(n-1)p(n+1)/p(n)^2$. Then for all $n \geq 2$, we have

$$4(1-u_n)(1-u_{n+1}) < \left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right)(1-u_nu_{n+1})^2.$$

Determinantal Inequalities of Partition Function

Definition

A sequence $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a multiplier sequence if, wherever the real polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ has only real zeros, the polynomial $\Gamma(P(x)) = \sum_{k=0}^{n} \gamma_k a_k x^k$ also has only real zeros.

A sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if its exponential generating function $P(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ belongs to $\mathcal{LPI}$.

Determinantal Inequalities of Partition Function

Theorem (Craven and Csordas, Pacific J. Math., 1989)

If $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k > 0$, is a multiplier sequence, then

$$\begin{vmatrix} \gamma_k & \gamma_{k+1} & \gamma_{k+2} \\ \gamma_{k-1} & \gamma_k & \gamma_{k+1} \\ \gamma_{k-2} & \gamma_{k-1} & \gamma_k \end{vmatrix} \geq 0, \ \text{for } k = 2, 3, 4, \ldots \tag{16}$$

Determinantal Inequalities of Partition Function

Theorem (Jia and Wang, Proc. Royal. Edinb. Soc., Section A, 2019)

Let $p(n)$ denote the partition function and

$$M_3(p(n)) = \begin{pmatrix} p(n) & p(n+1) & p(n+2) \\ p(n-1) & p(n) & p(n+1) \\ p(n-2) & p(n-1) & p(n) \end{pmatrix} \qquad (17)$$

Then for $n \geq 222$, we have

$$\det M_3(p(n)) > 0. \qquad (18)$$

Determinantal Inequalities of Partition Function

Conjecture (Jia and Wang, Proc. Royal. Edinb. Soc., Section A, 2019)

Let $M_k(p(n)) = (p(n-i+j))_{1\le i,j\le k}$. For any given k, there exists a positive integer $n(k)$ such that for $n > n(k)$,

$$\det M_k(p(n)) > 0. \tag{19}$$

Outline

Overpartition

An overpartition of a nonnegative integer n is a partition of n where the first occurrence of each distinct part may be overlined. For example, the eight overpartitions of 3 are

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

Let $\overline{p}(n)$ denote the number of overpartitions of n.

Overpartition

Hardy and Ramanujan stated that

$$\overline{p}(n) = \frac{1}{4\pi}\frac{d}{dn}\frac{e^{\pi\sqrt{n}}}{\pi\sqrt{n}} + \frac{\sqrt{3}}{2\pi}\cos\left(\frac{2}{3}n\pi - \frac{1}{6}\pi\right)\frac{d}{dn}\left(e^{\pi\sqrt{n}/3}\right) + \cdots + O\left(n^{-1/4}\right). \tag{20}$$

Zuckerman gave the following Rademacher-type convergent series.

$$\overline{p}(n) = \frac{1}{2\pi}\sum_{\substack{k\geq 1\\ 2\nmid k}}\sqrt{k}\sum_{\substack{0\leq h<k\\ (h,k)=1}}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-2\pi ihn/k}\frac{d}{dn}\left(\frac{\sinh(\pi\sqrt{n}/k)}{\sqrt{n}}\right), \tag{21}$$

where

$$\omega(h,k) = \exp\left(\pi i\sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k} - \left\lfloor\frac{hr}{k}\right\rfloor - \frac{1}{2}\right)\right). \tag{22}$$

Overpartition

Theorem (Engel, Ramanujan J., 2017)

The function $\overline{p}(n)$ is log-concave for $n \geq 2$.

Theorem (Wang, Xie and Zhang, Adv. Appl. Math., 2018)

For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,

$$0 < \triangle^r \overline{p}(n) \leq 2^{r-3}\left(1 - 2^{-\frac{3}{2}}\right)\zeta(3/2)\frac{e^{\pi\sqrt{n+r}}}{n+r}.$$

Overpartition

Theorem (Wang, Xie and Zhang, Adv. Appl. Math., 2018)

For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,

$$0 < (-1)^{r-1} \triangle^r \log \overline{p}(n) < \frac{\pi}{2} \left(\frac{1}{2}\right)_{r-1} \frac{1}{n^{r-\frac{1}{2}}}.$$

Theorem (Liu and Zhang, arXiv:1808.05091)

For $n > m > 1$, we have

$$\overline{p}(n)^2 - \overline{p}(n-m)\overline{p}(n+m) > 0.$$

For a, b are positive integers with $a, b > 1$, we have

$$\overline{p}(a)\overline{p}(b) > \overline{p}(a+b).$$

Overpartition

Theorem (Liu and Zhang, arXiv:1808.05091)

Let

$$u_n = \frac{\overline{p}(n-1)\overline{p}(n+1)}{\overline{p}(n)^2}.$$

For $n \geq 16$, we have

$$4(1-u_n)(1-u_{n+1}) > (1-u_n u_{n+1})^2.$$

Outline

Combinatorial Sequences

It is natural to consider whether other combinatorial sequences also satisfy this inequality. Unfortunately, many celebrated infinitely combinatorial sequences are log-convex, not log-concave, such as the Motzkin numbers, the Fine numbers, the central Delannoy numbers, the Franel numbers of order 3 and the Domb numbers. Thus they definitely do not satisfy the higher order Turán inequalities.

Combinatorial Sequences

Došlić introduced log-balanced sequences as follows: A sequence $\{a_n\}_{n\geq 0}$ of positive real numbers is log-balanced if $\{a_n\}_{n\geq 0}$ is log-convex and the sequence $\{\frac{a_n}{n!}\}_{n\geq 0}$ is log-concave.

Theorem (Došlić, Discrete Math., 2005)

The Motzkin numbers, the Fine numbers, the Franel number of order 3 *and* 4*, the Apéry numbers, the central Delannoy numbers and the large Schöder numbers are log-balanced.*

Combinatorial Sequences

Let a_n be a sequence satisfying the following three-term recurrence relations

$$a_{n+1} = u_n a_n + v_n a_{n-1}, \tag{23}$$

where u_n and v_n are rational functions in n and $u_n > 0$ for $n \geq 1$.

Combinatorial Sequences

To prove the sequence $\{\frac{a_n}{n!}\}_{n\geq 0}$ satisfies the higher order Turán inequalities, it suffices to prove

$$4\left(\frac{a_n^2}{n!^2}-\frac{a_{n-1}}{(n-1)!}\frac{a_{n+1}}{(n+1)!}\right)\left(\frac{a_{n+1}^2}{(n+1)!^2}-\frac{a_n}{n!}\frac{a_{n+2}}{(n+2)!}\right)-\left(\frac{a_n}{n!}\frac{a_{n+1}}{(n+1)!}-\frac{a_{n-1}}{(n-1)!}\frac{a_{n+2}}{(n+2)!}\right)^2>0. \tag{24}$$

Combinatorial Sequences

Find appropriate $g(n)$ and $h(n)$ satisfying that there exists a positive N such that for $n > N$,

(1) $g(n) < \frac{a_{n+1}}{a_n} < h(n)$,

(2) $f(g(n)) > 0$ and $f(h(n)) > 0$,

(3) $f(x)$ is monotone on the interval $[g(n), h(n)]$.

Combinatorial Sequences

Theorem (Wang, Adv. Appl. Math., 2019)

If a sequence $\{a_n\}_{n\geq 0}$ satisfying the above recurrence relation , and we can find $g(n)$ and $h(n)$ so that there exists a positive integer N such that

(1) $g(n) < \frac{a_{n+1}}{a_n} < h(n)$,

(2) $f(g(n)) > 0$ *and* $f(h(n)) > 0$,

(3) $f^{(i)}(g(n))$ *and* $f^{(i)}(h(n))$ *have the same signs for* $i = 1, 2, 3$,

then the sequence $\{\frac{a_n}{n!}\}_{n\geq 0}$ *satisfies the higher order Turán inequality.*

Combinatorial Sequences

For example,

$$\begin{aligned}
\frac{3(4n+3)(4n+7)}{16(n+1)(n+3)} &< \frac{M_{n+1}}{M_n} < \frac{3(2n+5)}{2(n+4)} \\
\frac{4(n+1)^2-2(n+1)+\frac{2}{3}}{(n+1)(n+2)} &< \frac{F_{n+1}}{F_n} < \frac{4n+6}{n+3} \\
\frac{8n^2+8n+\frac{16}{9}}{(n+1)^2} &< \frac{F^{(3)}_{n+1}}{F^{(3)}_n} < \frac{8n^2+8n+3}{(n+1)^2}.
\end{aligned}$$

Combinatorial Sequences

Theorem (Wang, Adv. Appl. Math., 2019)

The Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and the Domb numbers satisfy the higher order Turán inequality.

Theorem (Wang, Adv. Appl. Math., 2019)

the 3-th associated Jensen polynomial

$$g_n(x) = \sum_{k=0}^{3} \binom{n}{k} \frac{a_{n+k}}{(n+k)!} x^k$$

has only real zeros for the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 the Domb numbers.

Combinatorial Sequences

Conjecture (Wang, Adv. Appl. Math., 2019)

Let $\{a_n\}_{n\geq 0}$ are the sequences of the Motzkin numbers, the Fine numbers, the Franel number of order 3 and the Domb numbers. For any given integer $m \geq 4$, there exists a positive integer $N(m)$ such that for $n > N(m)$, the m-th associated Jensen polynomial

$$\sum_{i=0}^{m} \binom{m}{i} \frac{a_{n+i}}{(n+i)!} x^i$$

has only real zeros.

The End

Thank you for your attention