Derived Matroids

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Outline

1. Motivations and Backgrounds
2. Derived Matroids δM
3. Classification of Derived Sequences $\delta^k M$
4. Characterize $\delta M \cong M^*$

Motivation

Dependencies of Holes

In algebraic topology, the homology groups examine the independent holes of the topological spaces. What can we say about the dependence relations among these holes? **E.g.**, For 1-dimensional simplicial complexes (or graphs), it is about the dependencies among all cycles of graphs.

Rota's Questions

Dependencies of Circuits

At the Bowdoin College Summer 1971 NSF Conference on Combinatorics, Gian-Carlo Rota posed the following question: The minimal dependent sets of vectors in a space V may be regarded as vectors in the derived space δV over the same field by using the vectors of V as a basis for δV. Can this same sort of process be applied to the dependent sets of a matroid M to investigate the "**dependencies among dependencies**"? If so, what properties does δM, the derived matroid, posses?"

Longyear's Questions

Longyear answered Rota's first question in the case of binary matroids, and proposed the following questions:

- Question 1. (a) What effect does δ have on the flats of a matroid? (b) On the dual?
- Question 2. How many different (nonisomorphic) binary matroids M are there for which δM has rank r?
- Question 3. (a) When does $\delta M = M$? (b) When is there a matroid N for which $\delta N = M$? (c) If $\delta^{k+1} M = \delta(\delta^k M)$, when can $\delta^k M = \delta^j M$?
- Question 4. If M is $U_{1,3}$, then δM is $U_{2,3}$, $\delta^2 M$ is $U_{1,1}$ and $\delta^3 M$ is $U_{0,0}$. Characterize those M for which $\delta^k M$ can eventually be $U_{0,0}$.

Matroids

Circuit axioms

A matroid $M = (E, \mathscr{C})$ is an ordered pair of a finite set E and a collection $\mathscr{C}$ (called **circuits**) of subsets of E such that

(C1) $\emptyset \notin \mathscr{C}$.

(C2) If $I_1 \neq I_2 \in \mathscr{C}$, then $I_1 \nsubseteq I_2$.

(C3) If $I_1, I_2 \in \mathscr{C}$ are distinct circuits and $e \in I_1 \cap I_2$, then there exists a circuit $I_3 \in \mathscr{C}$ such that $I_3 \subseteq (I_1 \cup I_2) - e$.

Representable Matroids

- The matroid $M = (E, \mathscr{C})$ is **representable** over the field $\mathbb{F}$ if there is a vector space $V = \mathbb{F}^n$ and a representation $\varphi : E \to V$ satisfying that

$$I \in \mathscr{C} \quad \Leftrightarrow \quad \{\varphi(e) \mid e \in I\} \text{ is a minimal dependent set of } V.$$

- The pair (M, φ) denotes an $\mathbb{F}$-**represented matroid**.

Circuit Vectors

- (M, φ): $\mathbb{F}$-represented matroid;
- $E = \{e_1, \ldots, e_m\}$: the ground set of M;
- $\mathscr{C} = \mathscr{C}(M)$: the set of circuits of M.

Circuit Vectors

For each circuit $I \in \mathscr{C}(M)$, there exists a unique vector $\mathbf{c}_I = (c_1, \ldots, c_m) \in \mathbb{F}^m$ (up to a constant) such that

$$\sum_{i=1}^{m} c_i \varphi(e_i) = 0, \quad \text{where } \begin{cases} c_i \neq 0 & \text{for } e_i \in I, \\ c_i = 0 & \text{for } e_i \notin I. \end{cases}$$

We call $\mathbf{c}_I$ a **circuit vector** of (M, φ)

Derived Matroids

- Associated with the $\mathbb{F}$-represented matroid (M, φ), there is an $\mathbb{F}$-represented matroid $(\delta M, \delta\varphi)$ with ground set $\mathscr{C}(M)$, the set of circuits of M, such that

$$(\delta\varphi)(I) = \mathbf{c}_I \qquad \text{for } I \in \mathscr{C}(M).$$

Derived Matroid (Oxley-Wang, 2019+)

We call the $\mathbb{F}$-represented matroid $(\delta M, \delta\varphi)$ the **derived matroid** of (M, φ).

- $\delta U_{1,n} \cong M(K_n)$,
- $\delta U_{n-2,n} \cong U_{2,n}$. In particular, $\delta U_{2,4} \cong U_{2,4}$.

Derived Matroids

Derived Sequence (Oxley-Wang, 2019+)

Let $(\delta^0 M, \delta^0\varphi) = (M, \varphi)$. Inductively, for any positive integer k, the k**th derived matroid** $(\delta^k M, \delta^k\varphi)$ of M is the derived matroid of $(\delta^{k-1}M, \delta^{k-1}\varphi)$. The **derived sequence** of (M, φ) is the sequence

$$(\delta^0 M, \delta^0\varphi), (\delta^1 M, \delta^1\varphi), (\delta^2 M, \delta^2\varphi), \ldots.$$

- $\delta^k U_{2,4} \cong U_{2,4}$ for all $k \geq 0$.

- The following matrix A represents $M(K_4)$ over both $GF(2)$ and $GF(3)$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix}$$

  Write A_2 and A_3 for the interpretations of A over $GF(2)$ and $GF(3)$, respectively. Hence we can view $M[A_2]$ and $M[A_3]$ as $GF(2)$- and $GF(3)$-represented matroids.

- It is not difficult to check that

$$\delta M[A_2] \cong F_7 \quad \text{and} \quad \delta M[A_3] \cong F_7^-$$

  So $\delta M[A_3] \not\cong \delta M[A_2]$.

In contrast to the above, where we considered representations of a matroid over two different fields, if we fix the field $\mathbb{F}$, then the derived matroid of a binary or ternary matroid does not depend on the representation.

Theorem (Oxley-Wang, 2019+)

Let $\mathbb{F}$ be a field. Then, for all $\mathbb{F}$-represented matroids (M, φ) the derived matroid δM does not depend on the $\mathbb{F}$-representation φ if and only if $\mathbb{F}$ is $GF(2)$ or $GF(3)$.

Connected Matroids

- Given two matoids M and N on the ground sets E and F respectively. The **direct sum** $M \oplus N$ is the matroid $(E \sqcup F, \mathscr{C}(M) \sqcup \mathscr{C}(N))$. A matroid is **connected** if it is not isomorphic to a direct sum of two proper submatroids.

Structure Decomposition (Oxley-Wang, 2019+)

Let (M, φ) be an $\mathbb{F}$-represented matroid such that M has no coloops. If $M = M_1 \oplus M_2$, then $\delta M = \delta M_1 \oplus \delta M_2$. Conversely, if $\delta M = N_1 \oplus N_2$, then there are matroids M_1 and M_2 such that $M = M_1 \oplus M_2$ and $N_i = \delta M_i$ for each $i = 1, 2$.

Main Lemma

Lemma (Oxley-Wang, 2019+)

Let M be a nonempty connected matroid and $r^*(M) = r(M^*)$ the corank of M. Then

$$|\delta M| \geq \binom{r^*(M)+1}{2} \quad \text{and} \quad r^*(\delta M) \geq \binom{r^*(M)}{2}.$$

Corollary (Oxley-Wang, 2019+)

For a connected representable matroid M and $k \geq 0$, the matroid $\delta^k M$ is connected and

$$r^*(\delta^k M) \geq 2^k(r^*(M) - 3) + 3.$$

Cyclic Type

Answers of Longyear's Questions 3(a) and 3(c) for represented matroids over arbitrary fields.

Cyclic Type (Oxley-Wang, 2019+)

Let (M, φ) be a nonempty $\mathbb{F}$-represented matroid. If $\delta^k M \cong M$ for some $k \geq 1$, then M is a direct sum of matroids each of which is isomorphic to $U_{2,4}$.

Finite Type

Answers of Longyear's Questions 4 for represented matroids over arbitrary fields.

Finite Type (Oxley-Wang, 2019+)

Let M be a represented matroid such that $\delta^k M \cong U_{0,0}$ for some $k \geq 0$. Then $\delta^3 M \cong U_{0,0}$ and each component of M is isomorphic to one of the following matroids

1. $U_{1,1}$,
2. a circuit,
3. the cycle matroid of a theta graph.

Divergent Type

Theorem (Oxley-Wang, 2019+)

Let M be a connected represented matroid that is not isomorphic to $U_{0,0}$, $U_{1,1}$, a circuit, the cycle matroid of a theta graph, or a matroid whose cosimplification is $U_{2,4}$. Then, for all $k \geq 1$,

$$r^*(\delta^{k+1}M) > r^*(\delta^k M).$$

Moreover, unless $M \cong U_{1,4}$, we have

$$r^*(\delta^k M) \geq 2^{k-1} + 3.$$

In the exceptional case, $r^*(M) = 3 = r^*(\delta M)$, $r^*(\delta^2 M) = 4$, and

$$r^*(\delta^k M) \geq 2^{k-2} + 3 \quad \text{for} \quad k \geq 2.$$

Characterize $\delta M \cong M^*$

- A **basis** B of M is a maximal subset of E containing no circuits of M.
- The **dual matroid** M^* is a matroid with the ground set E such that

$$B \text{ is a basis of } M \quad \Leftrightarrow \quad E - B \text{ is a basis of } M^*.$$

Case: $r^*(M) \leq 2$ (Fu-Wang, 2019+)

Let M be a connected representable matroid. If $r^*(M) \leq 2$, then

$$\delta M \cong M^* \quad \Longleftrightarrow \quad M^* \cong U_{2,n}, \text{ or } U_{1,1}, \text{ or } \emptyset.$$

Characterize $\delta M \cong M^*$

Case: $r^*(M) \geq 3$ (Fu-Wang, 2019+)

Let M be a connected $\mathbb{F}$-representable matroid. If $r^*(M) = m - r \geq 3$, then $\delta M \cong M^*$ if and only if there is a finite field $\mathbb{F}_q$ such that

$$M^* \cong PG(m - r, \mathbb{F}_q),$$

where $PG(m - r, \mathbb{F}_q)$ denotes the projective geometry of $\mathbb{F}_q^{m-r}$, a matroid consisting of all 1-dimensional subspaces in $\mathbb{F}_q^{m-r}$.

Working Problems

Known Results

$$\delta M \cong M^* \quad \Leftrightarrow \quad M^* \cong PG(m-r, \mathbb{F}_q), \text{ or } U_{2,n}, \text{ or } U_{1,1}, \text{ or } \emptyset$$

$$L(M) \cong L^*(M) \quad \Leftrightarrow \quad M \cong PG(m-r, \mathbb{F}_q), \text{ or } U_{2,n}, \text{ or } U_{1,1}, \text{ or } \emptyset$$

Observation

$$L(\delta M) \cong L(M^*) \quad \Leftrightarrow \quad L(M^*) \cong L^*(M^*).$$

$$L(\delta M) \cong L(M^*) \quad \Rightarrow \quad L(\delta M) \cong L^*(M^*).$$

Problem: $L(\delta M) \cong L^*(M^*) \quad \Rightarrow \quad L(\delta M) \cong L(M^*)$?

Thank You