Multipartite entangled states and irredundant orthogonal arrays

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Content

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- Quantum information is an interdisciplinary subject of quantum mechanics and information science, which is closely related to many disciplines such as computer science and mathematics.
- The phenomenon of entanglement is considered to be one of the most striking features of quantum mechanics. Quantum entanglement has been widely applied in quantum information theory, such as quantum key distribution, quantum secure communication, superdense coding and teleportation.

- An important issue concerns the construction of genuinely multipartite entangled states. The methods are mainly from the field of quantum information.

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- An important issue concerns the construction of genuinely multipartite entangled states. The methods are mainly from the field of quantum information.
- In 2014, Goyeneche and Życzkowski established a link between the combinatorial notion of orthogonal arrays and t-uniform states. They pointed out that a special orthogonal array, called irredundant orthogonal array, is corresponding to a t-uniform state.[a]

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$$|\alpha\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_v \end{pmatrix} \text{ and } |\beta\rangle = \begin{pmatrix} b_1 \\ \vdots \\ b_v \end{pmatrix}.$$

Tensor product:

$$|\alpha\rangle \otimes |\beta\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_v \end{pmatrix} \otimes |\beta\rangle = \begin{pmatrix} a_1|\beta\rangle \\ \vdots \\ a_v|\beta\rangle \end{pmatrix}.$$

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- $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is a standard orthogonal basis of $(\mathbb{C}^2)^{\otimes 2}$.

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- $\{|i_1 i_2 \cdots i_k\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_k\rangle : (i_1 i_2 \cdots i_k) \in \mathbb{Z}_v^k\}$ is a standard orthogonal basis of $(\mathbb{C}^v)^{\otimes k}$.

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In a quantum system composed of k subsystems,

Quantum state

A Quantum state $|\psi\rangle$ of a system consisting of k qudits with v-dimensional can be defined as:

$$|\psi\rangle = \sum_{i_1,i_2,\cdots,i_k} a_{i_1 i_2 \cdots i_k} |i_1 i_2 \cdots i_k\rangle,$$

where $a_{i_1 i_2 \cdots i_k} \in \mathbb{C}$, $i_1, i_2, \cdots, i_k \in \mathbb{Z}_v$, and $\sum_{i_1,i_2,\cdots,i_k} |a_{i_1 i_2 \cdots i_k}|^2 = 1$.

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- $|\psi\rangle \in \mathbb{C}^v \otimes \mathbb{C}^v \otimes \cdots \otimes \mathbb{C}^v = (\mathbb{C}^v)^{\otimes k} \cong \mathbb{C}^{v^k}$.

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  Example:

$$|v\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

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$$\begin{aligned} |\psi_5\rangle = \quad & \tfrac{1}{4}(|00000\rangle + |01111\rangle + |02222\rangle + |03333\rangle \\ & +|10123\rangle + |11032\rangle + |12301\rangle + |13210\rangle \\ & +|20231\rangle + |21320\rangle + |22013\rangle + |23102\rangle \\ & +|30312\rangle + |31203\rangle + |32130\rangle + |33021\rangle). \end{aligned}$$

Known results on t-uniform states

when $t = \lfloor k/2 \rfloor$

- Scott, Facchi and Helwig constructed three kinds of AME$(k, 2)$ states for $k = 2, 3, 5$ and 6 by quantum additive codes, group acting and graph states, respectively. When $k = 4$ or $k \geq 7$, there doesn't exist AME$(k, 2)$ states.[a b c d]

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[a] Scott A J. Multipartite entanglement, quantum-error-correcting codes, and entangling power of quantum evolutions. Phys. Rev. A., 2004, 69: 052330.

[b] Facchi P. Multipartite entanglement in qubit systems. Rend. Lincei Mat. Appl., 2009, 20: 25-67.

[c] Helwig W. Absolutely maximally entangled qudit graph states. arXiv: quant-ph/1306.2879.

[d] Huber F, Gühne O, Siewert J. Absolutely maximally entangled states of seven qubits do not exist. Phys. Rev. Lett., 2017,118: 200502.

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- Goyeneche et al. constructed a kind of AME(k, p) states with $k = 3, 4$ for prime $p > 2$ by multiunitary matrix. [a]

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- Raissi et al. constructed a kind of AME(k, q) states with prime power q for any $q \geq k - 1$ by liner MDS codes.[c]

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- Pang et al. constructed a kind of 3-uniform states of $k = 8$ and $k \geq 12$ qudits with local dimension of non-prime power $d \geq 6$ by irredundant orthogonal array.[c]

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- Orthogonal arrays have been widely applied in experimental design, computer science, coding cryptography, network communication theory, software engineering and so on.

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An orthogonal array (OA) of size N, factor k, level v, strength t and index λ, denoted by $OA(N; t, k, v)$ or $OA_\lambda(t, k, v)$, is an $N \times k$ array with entries from a set V of v symbols, such that in every $N \times t$ sub-array, each t-tuple on V occurs exactly λ time.

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- $N = \lambda v^t$.
- When $\lambda = 1$, $OA_\lambda(t, k, v)$ is denoted as $OA(t, k, v)$.

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- An $OA(t,k,v)(k \geq 2t)$ is actually an $IrOA(t,k,v)$.
- An $IrOA_\lambda(t,k,v)$ exists only if $\lambda \leq v^{k-2t}$.

Example

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$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

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It is also an $IrOA(8;2,6,2)$.

Relationship between OAs and t-uniform states

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Relationship between OAs and t-uniform states

Parameters correspondence

	Orthogonal array	Multipartite quantum state $\|\psi\rangle$
N	Runs	Number of linear terms in the state
k	Factors	Number of qudits
v	Levels	Dimension of the subsystem ($v = 2$ for qubits)
t	Strength	Class of entanglement (t-uniform)

Known results on OA

$t = 2, k = 5, 6$

$t = 3, k = 5, 6$

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- If $v \notin \{2, 3, 4, 6, 10, 22\}$, then an $OA(2, 6, v)$ exists.

$t = 3, k = 5, 6$

[a] Ji L. and Yin J. Construction of new orthogonal arrays and covering arrays of strength three. *J. Combin. Theory Ser.A*, 2010, **117**: 236-247.

Known results on OA

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- Let $v \geq 4$. If $v \not\equiv 2 \pmod 4$, then an $OA(3, 5, v)$ exists.
- Let v satisfy $gcd(v, 4) \neq 2$ and $gcd(v, 18) \neq 3$. Then there is an $OA(3, 6, v)$. Besides, an $OA(3, 6, 3u)$ with $u \in \{5, 7\}$ exists.[a]

[a] Ji L. and Yin J. Construction of new orthogonal arrays and covering arrays of strength three. *J. Combin. Theory Ser.A*, 2010, **117**: 236-247.

Known results on OA

- If q is a prime power and $t < q$, then an $OA(t, q+1, q)$ exists. Moreover, if $q \geq 4$ is a power of 2, an $OA(3, q+2, q)$ exists.

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- Suppose that $v = q_1 q_2 \ldots q_s$ is a standard factorization of v into distinct prime powers. If $q_i > t$, then an $OA(t, k+1, v)$ exists, where $k = min\{q_i : 1 \leq i \leq s\}$. [a]

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Research contents

In this talk, we mainly construct some kinds of 2-uniform states of $k = 5, 6$ quidts with any local dimension and $k = q, q + 1$ quidts with local dimension q, where q is a prime power.

i.e., $IrOA_\lambda(2, k, v)$, where $k = 5, 6$ with $\lambda \leq v$, $\lambda \leq v^2$ respectively;

$IrOA_\lambda(2, k, q)$, where $k = q, q + 1$ with $\lambda \leq v^{q-4}$.

Auxiliary designs

Row-divisible OAs

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Let A_1 be an $IrOA_{\lambda_1}(t, k, v)$ and A_2 be an $IrOA_{\lambda_2}(t, k, v)$ over the same set V. A_1 and A_2 are compatible if their superimposition constitutes an $IrOA_{\lambda_1+\lambda_2}(t, k, v)$ over V.

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- A set of ω $IrOA$s over the same set V are termed compatible if they are pairwise compatible.

Construction methods

Construction 1 (Superposing)

For any non-negative integers $m_1, m_2, \ldots, m_r$,

$$\mu_i - row - divisible\ OA_{\lambda_i}(t, k, v), 1 \leq i \leq r$$

$$\Rightarrow \sum_{i=1}^{r} m_i \mu_i - row - divisible\ OA_{\sum_{i=1}^{r} m_i \lambda_i}(t, k, v).$$

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Construction 2 (Weighting)

Let μ_1, μ_2 be two positive integers,

$$\left.\begin{array}{l} \mu_1 - row - divisible\ OA_{\lambda_1}(t,k,v_1) \\ \mu_2 - row - divisible\ OA_{\lambda_2}(t,k,v_2) \end{array}\right\} \Rightarrow \begin{array}{c} \mu_1\mu_2 - row - divisible \\ OA_{\lambda_1\lambda_2}(t,k,v_1v_2). \end{array}$$

Construction methods

Construction 3

Let v_1 and v_2 be two positive integers; $m_1, m_2, \ldots, m_r$ be r non-negative integers; and $\mu_1, \mu_2, \ldots, \mu_r$,$\lambda_1, \lambda_2, \ldots, \lambda_r$ be $2r$ positive integers. Moreover they satisfies $m_1\mu_1 + m_2\mu_2 + \cdots + m_r\mu_r \leq \mu$. Then

$$\left.\begin{array}{l} \mu \ compatible \ IrOA_{\eta}(t,k,v_2)s \\ \mu_i - row - divisible \ OA_{\lambda_i}(t,k,v_1) \end{array}\right\} \Rightarrow IrOA_{\eta \sum_{i=1}^{r} m_i\lambda_i}(t,k,v_1v_2).$$

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Construction 4

If an $OA(k-t, 2(k-t), v)$ exists, then v^{k-2t} compatible $OA(t,k,v)$s exist, so does an $IrOA_{\lambda}(t,k,v)$ for any $\lambda \leq v^{k-2t}$.

Construction methods

Corollary

Let v_1 and v_2 be two positive integers; m_1, m_2 be two non-negative integers; and μ_1, μ_2, λ_1, λ_2 be four positive integers. Moreover they satisfies $m_1\mu_1 + m_2\mu_2 \leq \mu$. Then

$$\left.\begin{array}{l} \mu \ compatible \ IrOA(t,k,v_2)s \\ \mu_1 - row - divisible \ OA_{\lambda_1}(t,k,v_1) \\ \mu_2 - row - divisible \ OA_{\lambda_2}(t,k,v_1) \end{array}\right\} \Rightarrow IrOA_{\eta(m_1\lambda_1 + m_2\lambda_2)}(t,k,v_1v_2).$$

Main results for $t = 2$ and $k = 5$

Theorem 1

If v be a positive integer which satisfies $gcd(v, 4) \neq 2$ and $gcd(v, 18) \neq 3$, then an $IrOA_\lambda(2, 5, v)$ exists for any $\lambda \leq v$.

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Theorem 2

(1) Let $v = 2m$, where $(m, 2) = 1$ and $gcd(m, 18) \neq 3$, then an $IrOA_\lambda(2, 5, v)$ exists for any $2 \leq \lambda \leq \frac{1}{2}v - 1$.

(2) Let $v = 3m$, where $(m, 3) = 1, m \geq 5$, and $gcd(m, 4) \neq 2$, then an $IrOA_\lambda(2, 5, v)$ exists for any λ with even $2 \leq \lambda \leq \frac{2}{3}v$ and odd $3 \leq \lambda \leq \frac{2}{3}v - 3$.

(3) Let $v = 6m$, where $(m, 6) = 1$, then an $IrOA_\lambda(2, 5, v)$ exists for any λ with even $2 \leq \lambda \leq \frac{2}{3}v - 2$ and odd $3 \leq \lambda \leq \frac{2}{3}v - 11$.

Main results for $t = 2$ and $k = 6$

Theorem 3

If $v = q_1 q_2 \ldots q_s$ is a standard factorization of v into distinct prime powers where $q_i \geq 7$, $1 \leq i \leq s$, then an $IrOA_\lambda(2, 6, v)$ exists for any $\lambda \leq v^2$.

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If $v = q_1 q_2 \dots q_s$ is a standard factorization of v into distinct prime powers where $q_i \geq 7$, $1 \leq i \leq s$, then an $IrOA_\lambda(2, 6, v)$ exists for any $\lambda \leq v^2$.

Theorem 4

Let $v = sm$, $(m, s) = 1$, and m with a standard prime powers factorization of $q_1 q_2 \dots q_l$ satisfying $q_i \geq 7$ for $1 \leq i \leq l$.

(1) when $s \in \{2, 4, 6, 10, 12, 20, 30\}$, an $IrOA_\lambda(2, 6, v)$ exists for any λ with even $2 \leq \lambda \leq \frac{2}{s^2} v^2$ and odd $3 \leq \lambda \leq \frac{2}{s^2} v^2 - 3$.

(2) when $s \in \{3, 5, 15, 60\}$, an $IrOA_\lambda(2, 6, v)$ exists for any λ with $2 \leq \lambda \leq \frac{2}{s^2} v^2$.

Main results for $t = 2$ and $k = q$, $q + 1$

Theorem 5

Let q be a prime power with $q > 3$, then there exist an $IrOA_\lambda(2, q, q)$ and an $IrOA_\lambda(2, q + 1, q)$ for any λ with $\lambda \leq q^2 - 5q + 5$.

3-uniform states

For 3-uniform states, we mainly discuss the existence of $k \geq 8$ qubits.

i.e., $IrOA(N; 3, k, 2)$, where $k \geq 8$ except for 9.

Hadamard matrix

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A Hadamard matrix of order n is an $n \times n$ matrix H_n of $+1$'s and -1's whose rows are orthogonal, i.e. which satisfies

$$H_n H_n^{\mathrm{T}} = nI_n,$$

where I_n is the $n \times n$ identity matrix, and H_n^{T} is the transpose of H_n.

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- A Hadamard matrix is called normalized, if its every entry in the first row or column is "1".

3-uniform states

Construction 5

If H_n is a normalized Hadamard matrix of order n, then

$$\begin{pmatrix} H_n \\ -H_n \end{pmatrix}$$

is an orthogonal array $OA(2n,3,n,2)$ of $+1$'s and -1's. Moreover, if al-l elements equaling to -1s are replaced with 0s, then an orthogonal array $OA(2n,3,n,2)$ of 1's and 0's exists.

Main results

There exists a 3-uniform state of 9 qubits.

$$
\begin{aligned}
|\phi_9\rangle = \; & -|000000000\rangle - |000001111\rangle + |000110011\rangle + |000111100\rangle \\
& + |001010101\rangle + |001011010\rangle - |001100110\rangle - |001101001\rangle \\
& - |010010110\rangle - |010011001\rangle + |010100101\rangle + |010101010\rangle \\
& + |011000011\rangle + |011001100\rangle - |011110000\rangle - |011111111\rangle \\
& + |100010111\rangle + |100011000\rangle + |100100100\rangle + |100101011\rangle \\
& + |101000010\rangle + |101001101\rangle + |101110001\rangle + |101111110\rangle \\
& + |110000001\rangle + |110001110\rangle + |110110010\rangle + |110111101\rangle \\
& + |111010100\rangle + |111011011\rangle + |111100111\rangle + |111101000\rangle .
\end{aligned}
$$

Main results

Theorem 6

There exists a 3-uniform state of k qubits for every $k \geq 8$.

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Further problems

- **Problem 1:** Find new methods, construct some IrOAs of strength 2, with large index λ.

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- **Problem 1:** Find new methods, construct some IrOAs of strength 2, with large index λ.
- **Problem 2:** Construct more IrOAs of strength $t \geq 3$ with any $1 < \lambda \leq v^{k-2t}$.

Thank you

for your attention!