Height and Saturation Level of Random Digital Trees

(joint with M. Drmota, H.-K. Hwang and R. Neininger)

Michael Fuchs

Department of Mathematical Sciences
National Chengchi University

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NATIONAL CHENGCHI UNIVERSITY

August 21st, 2019

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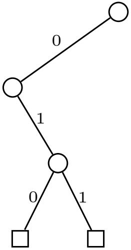

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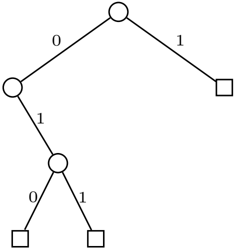

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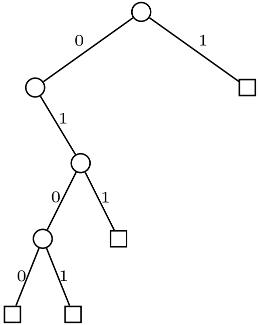

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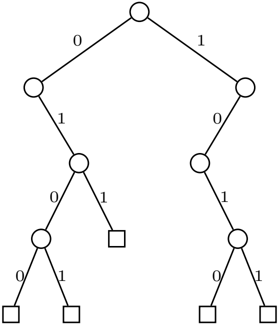

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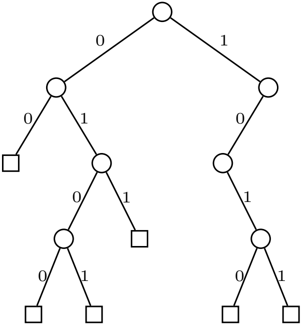

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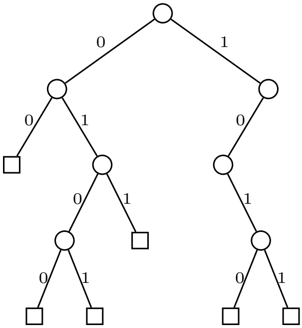

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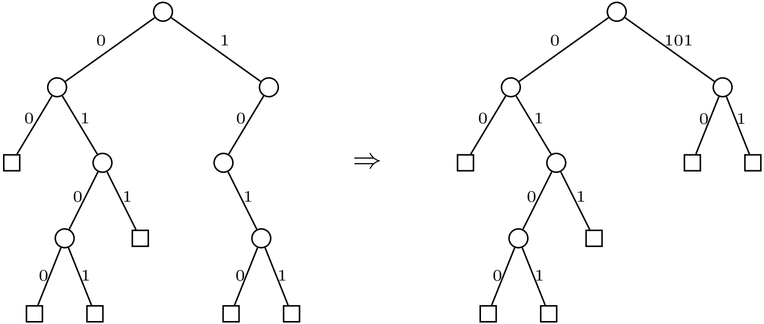

Digital Search Trees (DSTs)

Proposed by Edward G. Coffman and James Eve in 1970.

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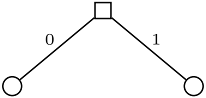

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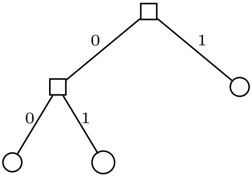

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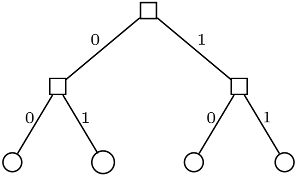

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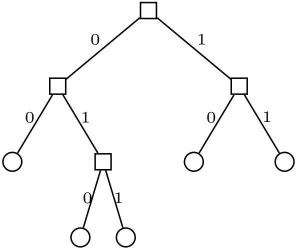

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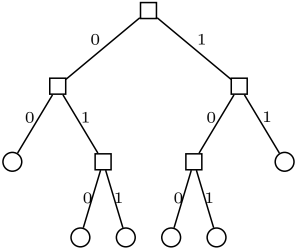

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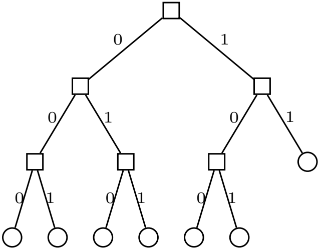

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- $p = 1/2$: *symmetric digital trees;*
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Question: What can be said about the "shape" of the tree?

This question is important because its answer will shed light on the complexity of algorithms performed on digital trees.

Three Shape Parameters

$H_n =$ longest path to a leaf;

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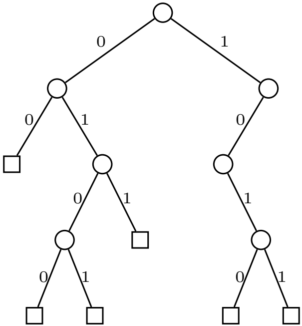

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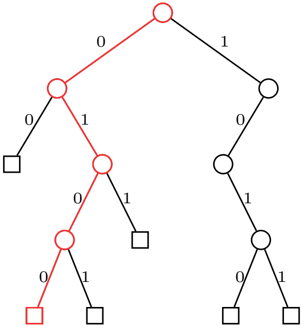

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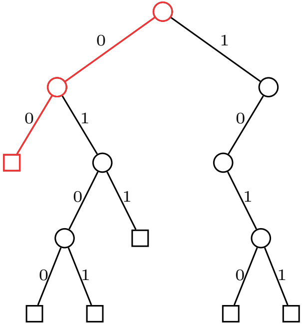

$H_n = 4;$

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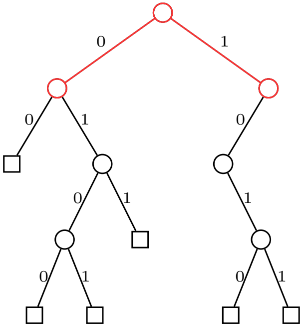

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Results for Tries (i)

Flajolet (1983):

Theorem

For symmetric tries,

$$\mathbb{P}(H_n \le k) \to e^{-e^{-t}},$$

where k *and* n *tend to infinity such that* $\log(2^{k+1}/n^2) \to t$.

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The above result was generalized to asymmetric tries by Pittel (with a probabilistic approach) and Jacquet & Règnier (with a complex-analytic approach) in 1986.

Results for Tries (ii)

Theorem (Pittel; 1986)

Let $p \geq q$. The distribution of S_n is concentrated on two points:

$$\mathbb{P}(S_n = k_S \text{ or } k_S + 1) \longrightarrow 1, \qquad \text{as } n \longrightarrow \infty.$$

Here, k_S is a sequence of n which satisfies

$$k_S = \begin{cases} \log_2 n - \log_2 \log n + \mathcal{O}(1), & \text{if } p = q; \\ \log_{1/q} n - \log_{1/q} \log \log n + \mathcal{O}(1), & \text{if } p \neq q. \end{cases}$$

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Theorem (Hwang & Nicodème & Park & Szpankowski; 2006)

We have,

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External and Internal Node Profile

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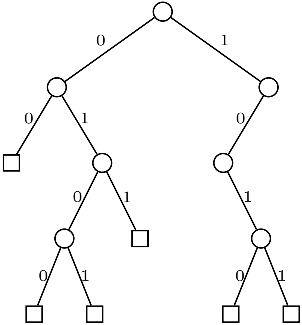

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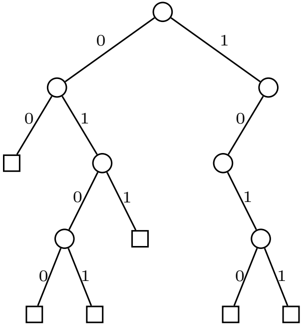

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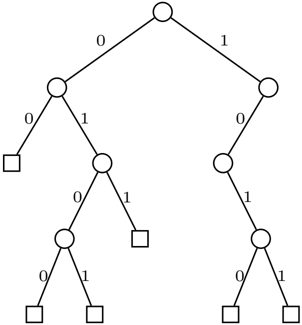

$B_{6,0} = 0;$ $\quad I_{6,0} = 1;$

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$B_{6,4} = 4;$ $\quad I_{6,4} = 0;$

H_n, S_n, F_n and the Profile of Tries

$$H_n = \max\{k \ : \ B_{n,k} > 0\};$$

$$S_n = \min\{k \ : \ B_{n,k} > 0\};$$

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So, for instance, we have

$$S_n > k \qquad \Longrightarrow \qquad B_{n,k} = 0$$

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and thus

$$\mathbb{P}(S_n > k) \leq \mathbb{P}(B_{n,k} = 0) \qquad \text{and} \qquad \mathbb{P}(S_n < k) \leq \sum_{\ell=0}^{k-1} \mathbb{P}(B_{n,\ell} > 0).$$

First and Second Moment Method

Theorem

Let X be a non-negative, integer-valued random variable. Then,

$$\mathbb{P}(X > 0) \le \mathbb{E}(X).$$

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$$\mathbb{P}(X = 0) \le \frac{\mathrm{Var}(X)}{(\mathbb{E}(X))^2}.$$

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Thus,

$$\mathbb{P}(S_n > k) \le \frac{\mathrm{Var}(B_{n,k})}{(\mathbb{E}(B_{n,k}))^2}$$

and

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Profile of Tries (Hwang et al.; 2006)

Let $p \geq q$ and

$$\alpha_1 := \frac{1}{\log(1/q)},\ \alpha_2 := \frac{p^2+q^2}{p^2\log(1/p)+q^2\log(1/q)},\ \alpha_3 := \frac{2}{\log(1/(p^2+q^2))}$$

and

$$\rho := \frac{1}{\log(p/q)}\log\left(\frac{1-\alpha\log(1/p)}{\alpha\log(1/q)-1}\right) \qquad \text{with } \alpha = \lim_n (k/\log n).$$

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Then,

$$\frac{\log \mathbb{E}(B_{n,k})}{\log n} \to \begin{cases} 0, & \text{if } \alpha \leq \alpha_1; \\ -\rho+\alpha\log(p^{-\rho}+q^{-\rho}), & \text{if } \alpha_1 \leq \alpha \leq \alpha_2; \\ 2+\alpha\log(p^2+q^2), & \text{if} \alpha_2 \leq \alpha \leq \alpha_3; \\ 0, & \text{if } \alpha \geq \alpha_3 \end{cases}$$

and $\mathrm{Var}(B_{n,k}) = \Theta(\mathbb{E}(B_{n,k}))$.

Concentration of Saturation Level and Height

Saturation Level:

Trees	$p = q$?	Concentration	Reference
Tries	$0 < p < 1$	2 points	HNPS2006
DSTs	$p = \frac{1}{2}$ $p \neq \frac{1}{2}$	2 points at most 3 points	DFHN2019+ DF2019+
PATRICIA Tries	$0 < p < 1$	?	?

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Height:

Trees	$p = q$?	Concentration	Reference
Tries	$0 < p < 1$	no	F1983; P1986; JR1986
DSTs	$p = \frac{1}{2}$ $p \neq \frac{1}{2}$	2 points ?	DFHN2019+ DF2019+
PATRICIA Tries	$p = \frac{1}{2}$ $p \neq \frac{1}{2}$	3 points ?	Conjectured by KS2002 ?

Profile of Symmetric DSTs: Mean

Let

$$Q(z) = \prod_{\ell=1}^{\infty} \left(1 - z2^{-\ell}\right), \qquad Q_n = \prod_{\ell=1}^{n} \left(1 - 2^{-\ell}\right) = \frac{Q(2^{-n})}{Q(1)}.$$

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Theorem (Drmota & F. & Hwang & Neininger; 2019+)

We have,

$$\mathbb{E}(B_{n,k}) = 2^k F(n/2^k) + \mathcal{O}(1),$$

where $F(x)$ is the positive function

$$F(x) = \sum_{j\geq 0} \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j Q(1)} e^{-2^j x}.$$

Profile of Symmetric DSTs: $F(x)$ (i)

As $x \to \infty$,

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As $x \to \infty$,

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and as $x \to 0$,

$$F(x) \sim \frac{X^{1/\log 2}}{\sqrt{2\pi x}} \exp\left(-\frac{(\log X \log X)^2}{2\log 2} - \sum_{j\in\mathbb{Z}} c_j (X \log X)^{-\chi_j} \right),$$

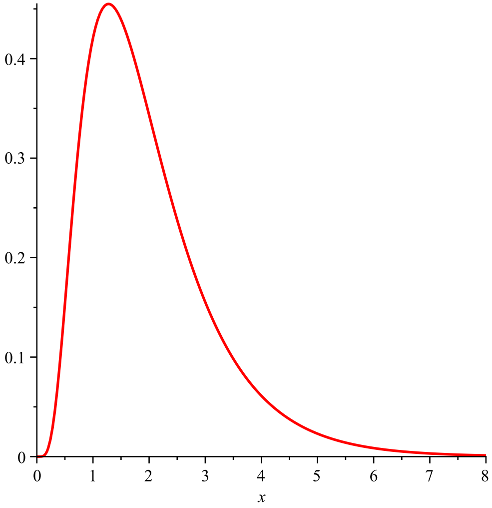
where $X = 1/(x\log 2)$, $\chi_j = 2j\pi i/\log 2$,

$$c_0 = \frac{\log 2}{12} + \frac{\pi^2}{6\log 2}$$

and

$$c_j = \frac{1}{2j\sinh(2j\pi^2/\log 2)}, \qquad (j \neq 0).$$

Profile of Symmetric DSTs: $F(x)$ (ii)



Profile of Symmetric DSTs: Variance

Theorem (Drmota & F. & Hwang & Neininger; 2019+)

We have,

$$\mathrm{Var}(B_{n,k}) = 2^k G(n/2^k) + \mathcal{O}(1),$$

where $H(x)$ *is a function with*

$$H(x) = \frac{e^{-x}}{Q(1)} + \mathcal{O}(xe^{-2x}), \qquad (x \to \infty)$$

and

$$H(x) \sim 2F(x), \qquad (x \to 0).$$

Profile of Symmetric DSTs: $G(x)$ (i)

We have,

$$G(x)=\sum_{j,r=0}^{\infty}\sum_{0\le h,\ell\le j}\frac{2^{-j}(-1)^{r+h+\ell}2^{-\binom{r}{2}-\binom{h}{2}-\binom{\ell}{2}+2h+2\ell}}{Q_rQ(1)Q_hQ_{j-h}Q_\ell Q_{j-\ell}}\varphi(2^{r+j},2^h+2^\ell;x),$$

where

$$\varphi(u,v;x)=\begin{cases}\dfrac{e^{-ux}-((v-u)x+1)e^{-vx}}{(v-u)^2}, & \text{if } u\neq v;\\ x^2e^{-ux}/2, & \text{if } u=v.\end{cases}$$

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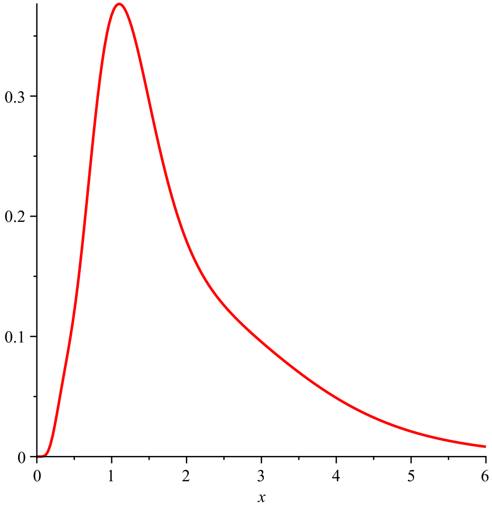
Proposition (Drmota & F. & Hwan & Neininger; 2019+)

$G(x)$ is a positive function on $(0, \infty)$.

Profile of Symmetric DSTs: $G(x)$ (ii)



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- Theory of Poisson Variance

  Developed by F., Hwang, Zacharovs (2010,2014).

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Profile of Symmetric DSTs: Limit Laws

$$k_f := \log_2 n - \log_2 \log n + 1 + \frac{\log_2 \log n}{\log n};$$

$$k_h := \log_2 n + \sqrt{2 \log_2 n} - \frac{1}{2} \log_2 \log_2 n + \frac{1}{\log 2} - \frac{3 \log \log n}{4\sqrt{2(\log n)(\log 2)}}.$$

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Theorem (Drmota & F. & Hwang & Neininger; 2019+)

(i) $\mathbb{E}(B_{n,k}), \mathrm{Var}(B_{n,k}) \to \infty$ *iff there exists* $\omega_n \to \infty$ *with*

$$k_f + \frac{\omega_n}{\log n} \le k \le k_h - \frac{\omega_n}{\sqrt{\log n}}.$$

(ii) *If* $\mathbb{E}(B_{n,k}) \to \infty$, *then*

$$\frac{B_{n,k} - \mathbb{E}(B_{n,k})}{\sqrt{\mathrm{Var}(B_{n,k})}} \xrightarrow{d} N(0,1).$$

Saturation Level and Height of Symmetric DSTs (i)

Recall,

$$\mathbb{E}(B_{n,k}) = 2^k F(n/2^k) + \mathcal{O}(1).$$

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$$\mathbb{E}(B_{n,k}) = 2^k F(n/2^k) + F'(n/2^k) - 2^{-k-1} n F''(n/2^k) + \mathcal{O}(n^{-1} + n/4^k)$$

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These results are sufficient!

Saturation Level and Height of Symmetric DSTs (ii)

Theorem (Drmota & F. & Hwang & Neininger; 2019+)

Let

$$k_H := \left\lfloor \log_2 n + \sqrt{2\log_2 n} - \frac{1}{2}\log_2\log_2 n + \frac{1}{\log 2} \right\rfloor.$$

Then, for the height H_n of symmetric DSTs,

$$\mathbb{P}(H_n = k_H \text{ or } k_H + 1) \longrightarrow 1, \qquad \text{as } n \longrightarrow \infty.$$

This was conjectured by Aldous & Shields (1988).

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Let $k_F := \lceil \log_2 n - \log_2 \log n \rceil$. Then, for the saturation level F_n of symmetric DSTs,

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Profile of Asymmetric DSTs: Notation

Assume that $p \geq q$.

Set

$$\alpha_1 = \frac{1}{\log(1/q)}, \qquad \alpha_2 = \frac{1}{\log(1/p)}$$

and

$$\rho = \frac{1}{\log(p/q)} \log\left(\frac{1-\alpha\log(1/p)}{\alpha\log(1/q)-1}\right),$$

where

$$\alpha = \lim_{n\to\infty} \frac{k}{\log n}.$$

Moreover, set

$$v = -\rho + \alpha\log(p^{-\rho} + q^{-\rho}).$$

Profile of Asymmetric DSTs: Mean & Variance

Theorem (Drmota & Szpankowski; 2011)

If $(\alpha_1+\epsilon)\log n \le k \le (\alpha_2-\epsilon)\log n$, *then*

$$\mathbb{E}(B_{n,k}) \sim H_1\left(\rho; \log_{p/q} p^k n\right) \frac{p^\rho q^\rho (p^{-\rho}+q^{-\rho})}{\sqrt{2\pi\alpha}\log(p/q)} \cdot \frac{n^v}{\sqrt{\log n}},$$

where $H_1(\rho;x)$ *is a* 1*-periodic function.*

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Theorem (Kazemi & Vahidi-Asl; 2011)

If $(\alpha_1+\epsilon)\log n \le k \le (\alpha_2-\epsilon)\log n$, *then*

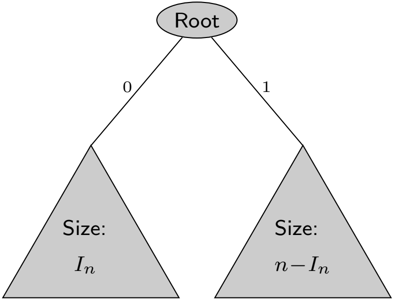
$$\mathrm{Var}(B_{n,k}) \sim H_2\left(\rho; \log_{p/q} p^k n\right) \frac{p^\rho q^\rho (p^{-\rho}+q^{-\rho})}{\sqrt{2\pi\alpha}\log(p/q)} \cdot \frac{n^v}{\sqrt{\log n}},$$

where $H_2(\rho; x)$ *is a* 1*-periodic function.*

Recurrences

$$B_{n+1,k} \stackrel{d}{=} B_{I_n,k-1} + B^*_{n-I_n,k-1}$$

- $I_n \stackrel{d}{=} \text{Binomial}(n,p)$;
- $B_{n,k} \stackrel{d}{=} B^*_{n,k}$;
- $B_{n,k}, B^*_{n,k}, I_n$ independent.



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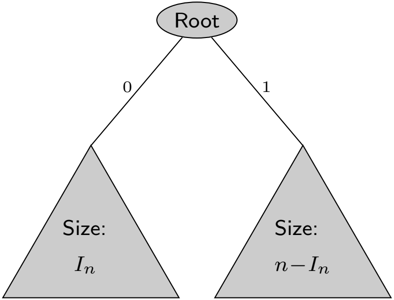

This gives the following recurrence for the mean ($\mu_{n,k} := \mathbb{E}(B_{n,k})$)

$$\mu_{n+1,k} = \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} \left(\mu_{j,k-1} + \mu_{n-j,k-1}\right).$$

Solving the Recurrence for the Mean (i)

$$\mu_{n+1,k} = \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} \left(\mu_{j,k-1} + \mu_{n-j,k-1}\right).$$

Solving the Recurrence for the Mean (i)

$$\mu_{n+1,k} = \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} \left(\mu_{j,k-1} + \mu_{n-j,k-1}\right).$$

- Consider the **Poisson-generating function**:

$$\tilde{f}_k(z) := e^{-z} \sum_{n} \mu_{n,k} \frac{z^n}{n!}.$$

Then,

$$\tilde{f}_k'(z) + \tilde{f}_k(z) = \tilde{f}_{k-1}(pz) + \tilde{f}_{k-1}(qz).$$

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- Consider the (normalized) **Mellin-transform**:

$$F_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty \tilde{f}_k(z) z^{s-1} \mathrm{d}s,$$

where $\Gamma(s)$ is the Gamma-function.

Solving the Recurrence for the Mean (ii)

Then,

$$F_k(s) - F_k(s-1) = T(s)F_{k-1}(s),$$

where

$$T(s) := p^{-s} + q^{-s}.$$

Solving the Recurrence for the Mean (ii)

Then,

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and by iteration

$$f(s,\omega) = \frac{g(s,\omega)}{g(0,\omega)}, \qquad g(s,\omega) := \prod_{j\geq 0} \frac{1}{1-\omega T(s-j)}.$$

Solving the Recurrence for the Mean (iii)

What is left to invert the whole process.

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- From $f(s,\omega)$ to $F_k(s)$:

$$F_k(s) = \frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{f(s,\omega)}{\omega^{k+1}}\mathrm{d}\omega,$$

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$$\tilde{f}_k(z) = \frac{1}{2\pi i}\int_{\mathcal{C}_2} \Gamma(s)F_k(s)z^{-s}\mathrm{d}s,$$

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- From $\tilde{f}_k(z)$ to $\mu_{n,k}$:

$$\mu_{n,k} = \frac{n!}{2\pi i}\int_{\mathcal{C}_3} \frac{e^z\tilde{f}_k(z)}{z^{n+1}}\mathrm{d}z$$

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Solving the Recurrence for the Mean (iv)

Drmota & Szpankowski (2011):

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- From $f(s,\omega)$ to $F_k(s)$ via residue theorem.
- From $F_k(s)$ to $\tilde{f}_k(z)$ and $\tilde{f}_k(z)$ to $\mu_{n,k}$ via saddle-point method.

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Drmota & F. (2019+):

$k \approx \alpha_1 \log n.$

Saddle point method for the inversion from $\tilde{F}_k(s)$ to $\tilde{f}_k(z)$ has to be replaced by the Poisson summation formula!

Profile of Asymmetric DSTs: Mean

Theorem (Drmota & F.; 2019+)

Let $k = \alpha_1(\log n - \log\log\log n + D)$, *where* $D = \mathcal{O}(1)$. *Then,*

$$\mathbb{E}(B_{n,k}) = \frac{1+o(1)}{\prod_{j\geq 1}(1-q^j)}(\log n)^{\frac{D-\log\log(p/q)-1}{\log(p/q)}} \times \left(\frac{(\log(1/q))^{-m_0}}{m_0!}(\log n)^{-\frac{H(m_0\log(p/q)-D+\log\log(p/q))}{\log(p/q)}} + \frac{(\log(1/q))^{-m_0-1}}{(m_0+1)!}(\log n)^{-\frac{H((m_0+1)\log(p/q)-D+\log\log(p/q))}{\log(p/q)}}\right) + \mathcal{O}\left((\log n)^{\frac{D-\log\log(p/q)-1}{\log(p/q)}}\right),$$

where $m_0 := \max(\lfloor(\frac{D-\log\log(p/q)}{\log(p/q)}\rfloor, 0)$ *and* $H(x) := e^x - 1 - x$.

Saturation Level of Asymmetric DSTs

Theorem (Drmota & F.; 2019+)

For the saturation level of asymmetric DSTs, we have

$$\mathbb{P}(F_n = k_F - 1 \quad \textit{or} \quad F_n = k_F \quad F_n = k_F + 1) \longrightarrow 1, \qquad \textit{as} \ \longrightarrow \infty,$$

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- Two point concentration holds for $p < 2/3$.

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Remarks:

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- We conjecture that two point concentration holds for $1/2 < p < 1$.
- We are currently working on a similar concentration result for the height.

Concentration of Saturation Level and Height

Saturation Level:

Trees	$p = q$?	Concentration	Reference
Tries	$0 < p < 1$	2 points	HNPS2006
DSTs	$p = \frac{1}{2}$ $p \neq \frac{1}{2}$	2 points at most 3 points	DFHN2019+ DF2019+
PATRICIA Tries	$0 < p < 1$	?	?

Height:

Trees	$p = q$?	Concentration	Reference
Tries	$0 < p < 1$	no	F1983; P1986; JR1986
DSTs	$p = \frac{1}{2}$ $p \neq \frac{1}{2}$	2 points ?	DFHN2019+ DF2019+
PATRICIA Tries	$p = \frac{1}{2}$ $p \neq \frac{1}{2}$	3 points ?	Conjectured by KS2002 ?

Profile of Asymmetric PATRICIA Tries

Theorem (Magner & Szpankowski; 2018)

If $(\alpha_1+\epsilon)\log n \le k \le (\alpha_2-\epsilon)\log n$, *then*

$$\mu_{n,k} \sim P_1\left(\rho; \log_{p/q} p^k n\right) \frac{p^\rho q^\rho (p^{-\rho}+q^{-\rho})}{\sqrt{2\pi\alpha}\log(p/q)} \cdot \frac{n^v}{\sqrt{\log n}},$$

and

$$\sigma_{n,k}^2 \sim P_2\left(\rho; \log_{p/q} p^k n\right) \frac{p^\rho q^\rho (p^{-\rho}+q^{-\rho})}{\sqrt{2\pi\alpha}\log(p/q)} \cdot \frac{n^v}{\sqrt{\log n}},$$

where $P_1(\rho;x)$ *and* $P_2(\rho;x)$ *are* 1*-periodic functions.*

Moreover,

$$\frac{B_{n,k}-\mu_{n,k}}{\sigma_{n,k}} \xrightarrow{d} N(0,1).$$

Saturation Level and Height of PATRICIA tries

By extending the previous study to the boundary.

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Theorem (Drmota & Magner & Szpankowski; 2019)

With high probability,

$$F_n = \begin{cases} \log_2 n - \log_2 \log n + o(\log\log n), & \textit{if } p = q; \\ \log_{1/q} n - \log_{1/q} \log\log n + o(\log\log\log n), & \textit{if } p > q. \end{cases}$$

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Theorem (Drmota & Magner & Szpankowski; 2019)

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$$H_n = \begin{cases} \log_2 n + \sqrt{2\log_2 n} + o(\sqrt{\log n}), & \text{if } p = q; \\ \log_{1/p} n + \frac{1}{2}\log_{p/q} \log n + o(\log\log n), & \text{if } p > q. \end{cases}$$

Summary and Open Problems

Profile of Random Digital Trees:

Trees	$p = q$?	Mean	Variance	CLT
Tries	$0 < p < 1$	✓	✓	✓
DSTs	$p = \frac{1}{2}$	✓	✓	✓
	$p \neq \frac{1}{2}$	✓	✓	?
PATRICIA Tries	$p = \frac{1}{2}$	?	?	?
	$p \neq \frac{1}{2}$	✓	✓	✓

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Major Open Tasks:

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PATRICIA Tries	$p=\frac{1}{2}$	?	?	?
	$p\neq\frac{1}{2}$	✓	✓	✓

Major Open Tasks:

- profile of symmetric PATRICIA tries;

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PATRICIA Tries	$p = \frac{1}{2}$	?	?	?
	$p \neq \frac{1}{2}$	✓	✓	✓

Major Open Tasks:

- profile of symmetric PATRICIA tries;
- refined results for the profile at the boundary of the "central range" for asymmetric PATRICIA tries (very complicated!).