Some new signed Euler-Mahonian identities and polynomials

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Outline

$\mathfrak{S}_n$:= the set of permutations of $\{1, 2, \ldots, n\}$

- $\mathfrak{S}_n = \langle s_1, s_2, \ldots, s_{n-1} \rangle$, where $s_i = (i\, i+1)$.
- $\ell(\pi)$:= the minimal number of generators needed to represent π
  - $41253 = (3\,4)(2\,3)(4\,5)(1\,2) = s_3 s_2 s_4 s_1$
  - $\ell(41253) = 4$
- number of inversions: $\mathsf{inv}(\pi) := |\{(i,j) : i < j \text{ and } \pi_i > \pi_j\}|$.
  - $\mathsf{inv}(41253) = 3 + 0 + 0 + 1 = 4$
- $\mathsf{inv}(\pi) = \ell(\pi)$ for $\pi \in \mathfrak{S}_n$.

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The descent number des and major index maj

$\mathsf{Des}(\pi) := \{i : \pi_i > \pi_{i+1}, i = 1, 2, \ldots, n-1\}$

- descent of π: $\mathsf{des}(\pi) := |\mathsf{Des}(\pi)|$ ▷ $\mathsf{des}(41253) = 2$
- major of π: $\mathsf{maj}(\pi) := \sum_{i \in \mathsf{Des}(\pi)} i$ ▷ $\mathsf{maj}(41253) = 1 + 4 = 5$

Theorem (MacMahon, 1913)

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathsf{maj}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\mathsf{inv}(\pi)}$$

- inv and maj are called *Mahonian* statistics (equi-distributed with ℓ)
- des is called *Eulerian* statistic

P.A. MacMahon, *The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects*, Amer. J. Math. 35 (1913) 281–322.

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Signed Euler-Mahonian identities

Theorem (Désarménien-Foata, 1992)

$$\sum_{\pi\in\mathfrak{S}_{2n}} (-1)^{\ell(\pi)} t^{\mathrm{des}(\pi)} = (1-t)^n \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}(\pi)}$$

$$\sum_{\pi\in\mathfrak{S}_{2n+1}} (-1)^{\ell(\pi)} t^{\mathrm{des}(\pi)} = (1-t)^n \sum_{\pi\in\mathfrak{S}_{n+1}} t^{\mathrm{des}(\pi)}$$

Theorem (Wachs, 1992)

$$\sum_{\pi\in\mathfrak{S}_{2n}} (-1)^{\ell(\pi)} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)} = \prod_{i=1}^{n}(1-tq^{2i-1}) \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}(\pi)} q^{2\mathrm{maj}(\pi)}$$

J. Désarménien and D. Foata, *The signed Eulerian numbers*, Discrete Math. 99 (1992), 49–58.

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Extend to Coxeter groups

- In $\mathfrak{S}_n$: Coxeter group of type A_{n-1}
  - Generalized by $\langle s_1, s_2, \ldots, s_{n-1} \rangle$, where $s_i = (i\, i+1)$
  - $\mathfrak{S}_n =$ permutations of $\{1, 2, \ldots, n\}$
    e.g. $\mathfrak{S}_2 = \{12, 21\}$
  - Signed Euler-Mahonian identity: $\sum_{\pi \in \mathfrak{S}_{2n}} (-1)^{\ell(\pi)} t^{\mathsf{des}(\pi)} q^{\mathsf{maj}(\pi)} = \cdots$
- In $\mathcal{B}_n$: Coxeter group of type B_n
  - Generalized by $\langle s_0, s_1, s_2, \ldots, s_{n-1} \rangle$, where $s_0 = (\bar{1}\, 1)$
  - $\mathcal{B}_n =$ signed permutations of $\{1, 2, \ldots, n\}$
    e.g. $\mathcal{B}_2 = \{12, 1\bar{2}, \bar{1}2, \bar{1}\bar{2}, 21, 2\bar{1}, \bar{2}1, \bar{2}\bar{1}\}$
  - $\ell_B = \mathsf{inv}(\pi) + \sum_{\pi_i < 0} |\pi_i|$
  - des??
  - maj??

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  - $\mathcal{B}_n$ = signed permutations of $\{1, 2, \ldots, n\}$
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  - $\ell_B = \mathsf{inv}(\pi) + \sum_{\pi_i < 0} |\pi_i|$
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Flag descent and major for $\mathcal{B}_n$

$\mathsf{Des}_F(\pi) := \{i : \pi_i > \pi_{i+1}\}$ w.r.t. $\bar{1} < \cdots < \bar{n} < 1 < \cdots < n$

- $\mathsf{fdes}(\pi) := 2 \cdot |\mathsf{Des}_F(\pi)| + \delta(\pi_i < 0)$
- $\mathsf{fmaj}(\pi) := 2 \cdot \sum_{i \in \mathsf{Des}_F(\pi)} + \mathsf{neg}(\pi)$
  - $\mathsf{Des}_F(\bar{3}1\bar{6}\bar{2}\bar{5}4) = \{2, 3\}$
  - $\mathsf{fdes}(\bar{3}1\bar{6}\bar{2}\bar{5}4) = 2 \cdot 2 + 1 = 5$
  - $\mathsf{fmaj}(\bar{3}1\bar{6}\bar{2}\bar{5}4) = 2 \cdot 5 + 4 = 14$

Theorem (Adin-Roichman, 2001)

$$\sum_{\pi \in \mathcal{B}_n} q^{\mathsf{fmaj}(\pi)} = \sum_{\pi \in \mathcal{B}_n} q^{\ell_B(\pi)}$$

R.M. Adin, Y. Roichman, *The flag major index and group actions on polynomial rings*, European J. Combin. 22 (2001) 431–446.

Signed Euler-Mahonian identities for $\mathcal{B}_n$

Theorem (Biagioli, 2006)

$$\sum_{\pi \in \mathcal{B}_{2n}} (-1)^{\mathsf{inv}_B(\pi)} q^{\mathsf{fmaj}(\pi)} = \prod_{i=1}^{n} \left(1 - q^{4i-2}\right) \sum_{\pi \in \mathcal{B}_n} q^{2\mathsf{fmaj}(\pi)}$$

Theorem (–, preprints)

$$\sum_{\pi \in \mathcal{B}_{2n}} (-1)^{\mathsf{inv}_B(\pi)} t^{\mathsf{fdes}(\pi)} q^{\mathsf{fmaj}(\pi)} = \prod_{i=1}^{n} \left(1 - t^2 q^{4i-2}\right) \sum_{\pi \in \mathcal{B}_n} t^{\mathsf{fdes}(\pi)} q^{2\mathsf{fmaj}(\pi)}$$

R. Biagioli, *Signed Mahonian polynomials for classical Weyl groups*, European J. Combin. 27 (2006) 207–217.

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Extend to 1-dim characters

- 1-dim character of G is a mapping $\chi : G \to \mathbb{C}$ being a homomorphism.
- 1-dim character χ of Coxeter groups:
  - For $\mathfrak{S}_n$: $\chi(\pi) = 1, (-1)^{\ell(\pi)}$
  - For $\mathcal{B}_n$: $\chi(\pi) = 1, (-1)^{\ell_B(\pi)}, (-1)^{\mathsf{neg}(\pi)}, (-1)^{\mathsf{inv}(|\pi|)}$

Signed Euler-Mahonian Polynomial

$$\sum_{\pi \in G} \chi(\pi) t^{\mathsf{stat}_1(\pi)} q^{\mathsf{stat}_2(\pi)}$$

- χ : any 1-dim character of G
- stat_1: Eulerian statistic
- stat_2: Mahonian statistic

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Main result 1: signed Euler-(Mahonian) identities for $\mathcal{B}_n$

Theorem (Wachs, 1992)

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Theorem (–, preprints)

- $$\sum_{\pi \in \mathcal{B}_{2n}} (-1)^{\ell_B(\pi)} t^{\mathsf{fdes}(\pi)} q^{\mathsf{fmaj}(\pi)} = \prod_{i=1}^{n} \left(1 - t^2 q^{4i-2}\right) \sum_{\pi \in \mathcal{B}_n} t^{\mathsf{fdes}(\pi)} q^{2\mathsf{fmaj}(\pi)}$$
- $$\sum_{\pi \in \mathcal{B}_{2n}} (-1)^{\mathsf{inv}(|\pi|)} t^{\mathsf{fdes}(\pi)} q^{\mathsf{fmaj}(\pi)} = \prod_{i=1}^{n} \left(1 - t^2 q^{4i-2}\right) \sum_{\pi \in \mathcal{B}_n} t^{\mathsf{fdes}(\pi)} q^{2\mathsf{fmaj}(\pi)}$$
- $$\sum_{\pi \in \mathcal{B}_n} (-1)^{\mathsf{neg}(\pi)} t^{\mathsf{fdes}(\pi)} q^{\mathsf{fmaj}(\pi)} = \sum_{\pi \in \mathcal{B}_n} t^{\mathsf{fdes}(\pi)} (-q)^{\mathsf{fmaj}(\pi)}$$

Main result 1: signed Euler-(Mahonian) identities for $\mathcal{B}_n$

Theorem (Désarménien-Foata, 1992)

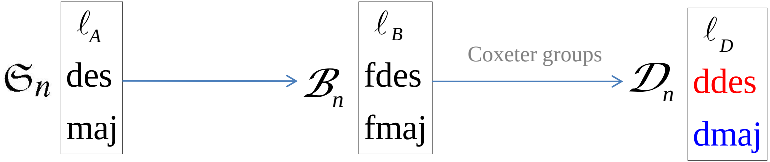
$$\sum_{\pi\in\mathfrak{S}_{2n+1}} (-1)^{\ell(\pi)} t^{\mathsf{des}(\pi)} = (1-t)^n \sum_{\pi\in\mathfrak{S}_{n+1}} t^{\mathsf{des}(\pi)}$$

Theorem (–, *preprints*)

- $$\sum_{\pi\in\mathcal{B}_{2n+1}} (-1)^{\mathsf{inv}(|\pi|)} t^{\mathsf{fdes}(\pi)} = (1-t^2)^n \sum_{\pi\in\mathcal{B}_{n+1}} t^{\mathsf{fdes}(\pi)}$$
- $$\sum_{\pi\in\mathcal{B}_{2n+1}} (-1)^{\ell_B(\pi)} t^{\mathsf{fdes}(\pi)} = (1-t^2)^n (1-t) \sum_{\pi\in\mathcal{B}_{n}} t^{2\cdot\mathsf{des}_B(\pi)}$$

- $\mathsf{des}_B(\pi) := |\{i : \pi_i > \pi_{i+1}, 0 \le i < n\}|$, where $\pi_0 := 0$

Extensions to Coxeter Groups of type D_n



Main result 2: signed Euler-(Mahonian) identities for $\mathcal{D}_n$

$\mathcal{D}_n$ = even-signed permutations of $\{1, 2, \ldots, n\}$

- Generalized by $\langle s_0', s_1, s_2, \ldots, s_{n-1} \rangle$, where $s_0' = (\bar{1}\,2)$
  - $\mathcal{D}_2 = \{12, \bar{1}\bar{2}, 21, \bar{2}\bar{1}\}$
- $\mathcal{D}_n$ has two 1-dim characters: 1 and $(-1)^{\ell_D}$
- $\mathsf{ddes}(\pi) := \mathsf{fdes}(\pi_1\pi_2 \cdots \pi_{n-1}|\pi_n|)$
- $\mathsf{dmaj}(\pi) := \mathsf{fmaj}(\pi_1\pi_2 \cdots \pi_{n-1}|\pi_n|)$

Theorem (–, *preprints*)

$$\sum_{\pi \in \mathcal{D}_{2n}} (-1)^{\ell_D} t^{\mathsf{ddes}(\pi)} q^{\mathsf{dmaj}(\pi)} = \prod_{i=1}^{n} (1 - t^2 q^{4i-2}) \sum_{\pi \in \mathcal{D}_n} t^{\mathsf{ddes}(\pi)} q^{2\mathsf{dmaj}(\pi)}$$

$$\sum_{\pi \in \mathcal{D}_{2n+1}} (-1)^{\ell_D(\pi)} t^{\mathsf{ddes}(\pi)} = (1 - t^2)^n \sum_{\pi \in \mathcal{D}_{n+1}} t^{\mathsf{ddes}(\pi)}$$

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$$\sum_{\pi\in\mathcal{D}_{2n}} (-1)^{\ell_D} t^{\mathsf{ddes}(\pi)} q^{\mathsf{dmaj}(\pi)} = \prod_{i=1}^{n}(1 - t^2q^{4i-2}) \sum_{\pi\in\mathcal{D}_n} t^{\mathsf{ddes}(\pi)} q^{2\mathsf{dmaj}(\pi)}$$

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Proof Sketch for even case: Step 1

For $\pi \in \mathcal{D}_{2n}$ let i be the smallest integer such that $2i-1$ and $2i$

1. have opposite signs,
2. are not in adjacent positions, or
3. are both at the last two positions with negative signs.

Let $\eta(\pi)$ be the even-signed permutation obtained from π by swapping the two letters $2i-1$ and $2i$.

e.g. $\eta(21\bar{\mathbf{3}}\bar{5}6\bar{\mathbf{4}}) = 21\bar{\mathbf{4}}\bar{5}6\bar{\mathbf{3}}$, $\quad \eta(\mathbf{2}\bar{\mathbf{1}}\bar{3}\bar{5}6\bar{4}) = \mathbf{1}\bar{\mathbf{2}}\bar{3}\bar{5}6\bar{4}$, $\quad \eta(21\bar{3}\bar{4}\bar{\mathbf{5}}\bar{\mathbf{6}}) = 21\bar{3}\bar{4}\bar{\mathbf{6}}\bar{\mathbf{5}}$

Fixed points $\mathcal{F}_{2n}$: letters $2i-1$ and $2i$ are adjacent and having the same sign, and both π_{2n-1} and π_{2n} are positive

- $\ell_D(\pi) = \ell_D(\pi') \pm 1$
- $\mathrm{Des}_F(\pi_1\pi_2\cdots\pi_{2n-1}|\pi_{2n}|) = \mathrm{Des}_F(\pi'_1\pi'_2\cdots\pi'_{2n-1}|\pi'_{2n}|)$

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Proof Sketch for even case: Step 2

Define a bijective correspondence $\phi : \mathcal{F}_{2n} \to \widehat{\mathcal{D}}_n$ as

1. Each pair of adjacent entries of type $\pm(2j-1), \pm 2j$ in $\mathcal{F}_{2n}$ is replaced by $\pm j$;
2. Each pair of adjacent entries of type $\pm 2j, \pm(2j-1)$ in $\mathcal{F}_{2n}$ is replaced by $\pm\hat{j}$;
3. After the two steps, if the number of negatives of the resulting permutation is odd, then change the sign of the last entry from positive to negative.

e.g. $\phi(\pi) = \phi(21\bar{5}\bar{6}8743) = \hat{1}\bar{3}\hat{4}\hat{\bar{2}} = \pi'$

- $(-1)^{\ell_D(\pi)} = (-1)^{|\mathsf{P}(\pi')|}$
- $\mathsf{ddes}(\pi) = \mathsf{ddes}(\pi') + 2|\mathsf{P}(\pi')|$
- $\mathsf{dmaj}(\pi) = 2\mathsf{dmaj}(\pi') + \sum_{i \in \mathsf{P}(\pi')}(4i-2)$

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Proof Sketch for even case: Step 3

$$\sum_{\pi\in\mathcal{D}_{2n}} (-1)^{\ell_D} t^{\mathsf{ddes}(\pi)} q^{\mathsf{dmaj}(\pi)}$$

$$\text{(Step 1)} = \sum_{\pi\in\mathcal{F}_{2n}} (-1)^{\ell_D} t^{\mathsf{ddes}(\pi)} q^{\mathsf{dmaj}(\pi)}$$

$$\text{(Step 2)} = \sum_{\pi'\in\widehat{\mathcal{D}}_n} \left((-1)^{|\mathsf{P}(\pi')|} t^{2|\mathsf{P}(\pi')|} q^{\sum_{i\in\mathsf{P}(\pi')}(4i-2)}\right) t^{\mathsf{ddes}(\pi')} q^{2\mathsf{dmaj}(\pi')}$$

$$= \sum_{\pi'\in\mathcal{D}_n} \left(\sum_{A\subseteq\{1,2,\ldots,n\}} (-1)^{|A|} t^{2|A|} q^{\sum_{i\in A}(4i-2)} \right) t^{\mathsf{ddes}(\pi')} q^{2\mathsf{dmaj}(\pi')}$$

$$= \prod_{i-1}^{n} (1 - t^2 q^{4i-2}) \sum_{\pi'\in\mathcal{D}_n} t^{\mathsf{ddes}(\pi')} q^{2\mathsf{dmaj}(\pi')}.$$

Proof Sketch for odd case: Step 1

For $\pi \in \mathcal{D}_{2n+1}$ let i be the smallest integer such that $2i-1$ and $2i$

1. are not in adjacent positions,
2. have opposite signs, and are not both at the last two positions, or
3. are both at the last two positions and $\pi_{2n} < 0$

Let $\iota(\pi)$ be the even-signed permutation obtained from π by swapping the two letters $2i-1$ and $2i$.

e.g. $\iota(\mathbf{5}8\bar{7}\bar{1}\bar{2}9\mathbf{6}3\bar{4}) = \mathbf{6}8\bar{7}\bar{1}\bar{2}9\mathbf{5}3\bar{4}$, $\quad \iota(58\bar{7}\bar{1}\bar{2}96\bar{\mathbf{3}}\mathbf{4}) = 58\bar{7}\bar{1}\bar{2}96\bar{\mathbf{4}}\mathbf{3}$,
$\iota(58\bar{7}\bar{1}\bar{2}96\bar{\mathbf{3}}\bar{\mathbf{4}}) = 58\bar{7}\bar{1}\bar{2}96\bar{\mathbf{4}}\bar{\mathbf{3}}$

Fixed points $\mathcal{F}_{2n+1}$:

- Letters $2i-1$ and $2i$ are adjacent.
- Letters $2i-1$ and $2i$ have the same sign if both of them are not at the last two positions.
- If $2i-1$ and $2i$ appear at the last two positions for some i, then $\pi_{2n} > 0$.

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- Letters $2i-1$ and $2i$ are adjacent.
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Proof Sketch for odd case: Step 2

Define a bijective correspondence $\phi : \mathcal{F}_{2n+1} \to \widehat{\mathcal{D}}_{n+1}$ as

1. Each pair of adjacent entries of type $\pm(2j-1), \pm 2j$ in $\mathcal{F}_{2n+1}$ but not at the last two positions is replaced by $\pm j$;
2. Each pair of adjacent entries of type $\pm 2j, \pm(2j-1)$ in $\mathcal{F}_{2n+1}$ but not at the last two positions is replaced by $\pm\hat{j}$;
3. The pair of entries of type $(2j-1), \pm 2j$ at the last two positions in $\mathcal{F}_{2n+1}$ is replaced by $\pm j$;
4. The pair of entries of type $2j, \pm(2j-1)$ at the last two positions in $\mathcal{F}_{2n+1}$ is replaced by $\pm\hat{j}$;
5. The entry $\pm(2n+1)$ in $\mathcal{F}_{2n+1}$ is replaced by $\pm(n+1)$;
6. After the above steps, if the number of negatives of the resulting permutation is odd, then change the sign of the last entry.

e.g. $\phi(21\bar{5}\bar{6}87\bar{9}4\bar{3}) = \hat{1}\bar{3}\hat{4}\bar{5}\hat{2}$ $\quad$ $\phi(21\bar{5}\bar{6}87439) = \hat{1}\bar{3}\hat{4}\hat{2}\bar{5}$

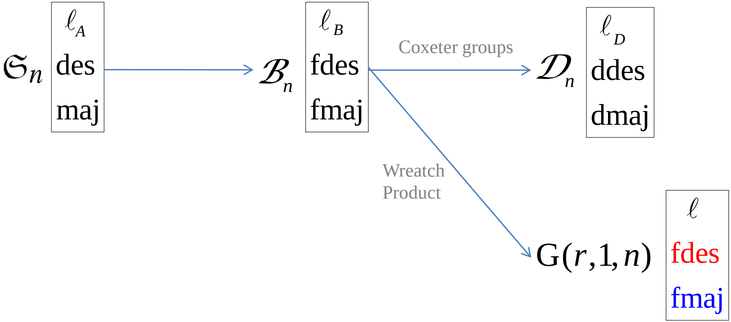
Proof Sketch for odd case: Step 3

- $(-1)^{\ell_D(\pi)} = (-1)^{|\mathsf{P}(\pi')|} = (-1)^{|\mathsf{L}(\pi')|}$
- $\mathsf{ddes}(\pi) = \mathsf{ddes}(\pi') + 2|\mathsf{L}(\pi')|$

$$\sum_{\pi \in \mathcal{F}_{2n+1}} (-1)^{\ell_D} t^{\mathsf{ddes}(\pi)} = \sum_{\pi' \in \widehat{\mathcal{D}}_{n+1}} (-1)^{|\mathsf{L}(\pi')|} t^{\mathsf{ddes}(\pi') + 2|\mathsf{L}(\pi')|}$$

$$= \sum_{\pi' \in \mathcal{D}_{n+1}} \left(\sum_{A \in [n]} (-1)^{|A|} t^{2|A|} \right) t^{\mathsf{ddes}(\pi')} = (1 - t^2)^n \sum_{\pi' \in \mathcal{D}_{n+1}} t^{\mathsf{ddes}(\pi')}$$

Extensions to Complex Reflection Groups $G(r, 1, n)$



$G(r, 1, n)$ and length function ℓ

$G(r, 1, n) := \mathbb{Z}r \wr \mathfrak{S}_n$: Wreath product
$= $ colored permutation group on $\{1, 2, \ldots, n\}$ with r colors

e.g. 1 color, 6 letters: $5\,6\,3\,1\,4\,2$
4 colors, 6 letters: $\bar{5}\,\bar{\bar{6}}\,\bar{3}\,\bar{\bar{\bar{1}}}\,4\,\bar{\bar{\bar{2}}}$

- $G(1, 1, n) = \mathfrak{S}_n$; $G(2, 1, n) = \mathcal{B}_n$.
- $G(r, 1, n) = \langle s_0, s_1, \ldots, s_{n-1} \rangle$ where $s_0 :=$ add one more bar on the first letter.
- $\ell(\pi) :=$ the minimal number of generators needed to represent π

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Represent χ in terms of ℓ

Theorem (–, *arXiv*)

$G(r,1,n)$ has $2r$ 1-dim characters

$$\chi_{a,b}(\pi) = (-1)^{a(\ell(\pi)-\sum z_i)}\omega^{b\sum z_i},$$

where $\omega = e^{2\pi i/r}$, $a = 0,1$ and $b = 0,1,\ldots,r-1$.

For $r = 2$, $\ell = \ell_B, \omega = -1$ and the four 1-dim characters are

- $\chi_{0,0} = (-1)^0(-1)^0 = 1$,
- $\chi_{0,1} = (-1)^0(-1)^{\sum z_i} = (-1)^{\mathsf{neg}(\pi)}$,
- $\chi_{1,0} = (-1)^{\ell(\pi)-\sum z_i}(-1)^0 = \cdots = (-1)^{\mathsf{inv}(|\pi|)}$,
- $\chi_{1,1} = (-1)^{\ell(\pi)-\sum z_i}(-1)^{\sum z_i} = (-1)^{\ell_B}$.

Flag descent and major $G(r, 1, n)$

$\mathsf{Des}_F(\pi) := \{i : \pi_i > \pi_{i+1}\}$ w.r.t.

$$1^{[r-1]} < \cdots < n^{[r-1]} < \cdots < \bar{1} < \cdots < \bar{n} < 1 < \cdots < n$$

- $\mathsf{fdes}(\pi) := r \cdot |\mathsf{Des}_F(\pi)| + z_1$
- $\mathsf{fmaj}(\pi) := r \cdot \sum_{i \in \mathsf{Des}_F(\pi)} + \mathsf{col}(\pi)$
  - $\mathsf{Des}_F(\bar{\bar{4}}\mathbf{5}\bar{1}\bar{\mathbf{3}}\bar{\mathbf{2}}\bar{\bar{\bar{6}}}) = \{2, 4, 5\}$
  - $\mathsf{fdes}(\bar{\bar{4}}\mathbf{5}\bar{1}\bar{\mathbf{3}}\bar{\mathbf{2}}\bar{\bar{\bar{6}}}) = 3 \cdot 3 + 2 = 9$
  - $\mathsf{fmaj}(\bar{\bar{4}}\mathbf{5}\bar{1}\bar{\mathbf{3}}\bar{\mathbf{2}}\bar{\bar{\bar{6}}}) = 3 \cdot 11 + 8 = 41$

R.M. Adin, Y. Roichman, *The flag major index and group actions on polynomial rings*, European J. Combin. 22 (2001) 431–446.

Main result 3

Theorem (–, *preprints*)

For $b = 0, 1, \cdots, r-1$, *we have*

$$\sum_{\pi \in G(r,1,2n)} \chi_{1,b}(\pi) t^{\mathrm{fdes}(\pi)} q^{\mathrm{fmaj}(\pi)} = \prod_{i=1}^{n} \left(1 - t^r q^{r(2i-1)}\right) \sum_{\pi \in G(r,1,n)} t^{\mathrm{fdes}(\pi)} (\omega^b q)^{2\mathrm{fmaj}(\pi)},$$

and

$$\sum_{\pi \in G(r,1,n)} \chi_{0,b}(\pi) t^{\mathrm{fdes}(\pi)} q^{\mathrm{fmaj}(\pi)} = \sum_{\pi \in G(r,1,n)} t^{\mathrm{fdes}(\pi)} (\omega^b q)^{\mathrm{fmaj}(\pi)}.$$

The case $G(r,1,2n+1)$ with $\chi_{1,b}$ for any b is missing!

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Concluding Remarks

Signed Euler-Mahonian identities

Generalize the following two identities.

Theorem (Wachs, 1992)

$$\sum_{\pi \in \mathfrak{S}_{2n}} (-1)^{\mathsf{inv}(\pi)} t^{\mathsf{des}(\pi)} q^{\mathsf{maj}(\pi)} = \prod_{i=1}^{n} (1 - tq^{2i-1}) \sum_{\pi \in \mathfrak{S}_n} t^{\mathsf{des}(\pi)} q^{2\mathsf{maj}(\pi)}$$

Theorem (Désarménien-Foata, 1992)

$$\sum_{\pi \in \mathfrak{S}_{2n+1}} (-1)^{\mathsf{inv}(\pi)} t^{\mathsf{des}(\pi)} = (1-t)^n \sum_{\pi \in \mathfrak{S}_{n+1}} t^{\mathsf{des}(\pi)}$$

Signed Euler-Mahonian identities

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$$\sum_{\pi\in W} \chi(\pi) t^{\mathsf{stat}_1} q^{\mathsf{stat}_2} = \cdots$$

- $W = \mathcal{B}_{2n}$: $(\mathsf{stat}_1, \mathsf{stat}_2) = (\mathsf{fdes}, \mathsf{fmaj})$
- $W = \mathcal{D}_{2n}$: $(\mathsf{stat}_1, \mathsf{stat}_2) = (\mathsf{ddes}, \mathsf{dmaj})$
- $W = G(r, 1, 2n)$: $(\mathsf{stat}_1, \mathsf{stat}_2) = (\mathsf{fdes}, \mathsf{fmaj})$

Signed Eulerian identities

Theorem (Désarménien-Foata, 1992)

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- $W = \mathcal{D}_{2n+1}$: $(\mathrm{stat}_1, \mathrm{stat}_2) = (\mathrm{ddes}, \mathrm{dmaj})$
- $W = G(r, 1, 2n+1)$: ???

More Future works

Let $G(r,p,n)$ denote the *complex reflection group* with parameters r,p,n, where $p|r$.

- $G(1,1,n) = \mathfrak{S}_n$, the Coxeter group of type A_{n-1}
- $G(2,1,n) = \mathcal{B}_n$, the Coxeter group of type B_n
- $G(2,2,n) = \mathcal{D}_n$, the Coxeter group of type D_n
- $G(r,1,n) = C_r \wr \mathfrak{S}_n$, the Wreath product of C_r with $\mathfrak{S}_n$

Question.

Is it possible to extend to $G(r,p,n)$?

Thank you for your attention