Weakly symmetric hexavalent graphs of order $9p$

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- Classification for half-arc-transitive case
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Let X be a regular graph and $\mathrm{Aut}(X)$ the full automorphism group.

Different types of transitivity

- **vertex-transitive**: $\mathrm{Aut}(X)$ is transitive on vertices.
- **edge-transitive**: $\mathrm{Aut}(X)$ is transitive on edges.
- **arc-transitive**: $\mathrm{Aut}(X)$ is transitive on arcs.
- **half-arc-transitive**: $\mathrm{Aut}(X)$ is transitive on vertices and edges but not on arcs.
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- **Such graphs with certain primitive action**: Praeger, Li, Fang and Lu, et al.
- **prime valency**: by using the structure of vertex stabilizers, and covering and lifting technique, for example, Feng, Marušič and Zhou, et al.
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Theorem 1.1, DM, 2017

Let p be a prime. Then any connected hexavalent arc-transitive graph of order $9p$ is isomorphic to one of the following graphs.

p	s-transitive	$\mathrm{Aut}(X)$	Num.
2	1-transitive	$(\mathrm{S}_3 \times \mathbb{Z}_3).D_{12}$	1
3	1-transitive	$\mathrm{Aut}(X)_v{=}D_{12},\ \mathrm{S}_4{\times}\mathbb{Z}_2,\ D_8{\times}\mathrm{S}_3$	4
5	1-transitive	$\mathbb{Z}_3.\mathrm{S}_6$	1
7	1-transitive	$G_2(2)$	2
p	1-transitive	$\mathrm{S}_3 \mathrm{\ wr\ } D_{6p}$	1
$p{\geq}3$	1-transitive	$\mathrm{S}_3 \mathrm{\ wr\ } D_{2p}$	1
$p{\equiv}1 (\mathrm{mod}\ 6)$	1-regular	$\mathbb{Z}_{9p} \rtimes \mathbb{Z}_6$	3
$p{\equiv}1 (\mathrm{mod}\ 6)$	1-regular	$(\mathbb{Z}_3^2 \times \mathbb{Z}_p) \rtimes \mathbb{Z}_6$	1

Ideas of proof

Let X be such graph, $A = \mathrm{Aut}(X)$ and N a minimal normal subgroup of A.

- X cannot be a normal Cayley graph on a non-abelian group.

  By using the automorphisms of non-abelian group of order $9p$.

- X cannot be 2-arc-transitive.

  By using the structure of vertex stabilizers of 2-arc-transitive hexavalent graphs, quotient graphs relative to the orbits of a minimal normal subgroup of A, and the K_3- and K_4-simple groups.

- N has two cases: $N_v \neq 1$ or $N_v = 1$.

  For $N_v \neq 1$, $X \cong C_{3p}[3K_1]$ or $C(3, p, 2)$.

  For $N_v = 1$, X is isomorphic to a normal Cayley graph on an abelian group of order $9p$ or some sporadic graphs.

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The half-arc-transitive case

Theorem 1.1, DM, 2019

Let p be a prime and X a connected hexavalent half-arc-transitive graph of order $9p$. Then X, the automorphism group $\mathrm{Aut}(X)$ and the vertex stabilizer $\mathrm{Aut}(X)_v$ for a vertex $v \in V(X)$ are described in the following table:

p	$\mathrm{Aut}(X)$	$\mathrm{Aut}(X)_v$	Numeration
$9 \mid (p-1)$	$\mathbb{Z}_{3p} \rtimes \mathbb{Z}_9$	$\mathbb{Z}_3$	1
$27 \mid (p-1)$	$\mathbb{Z}_p \rtimes \mathbb{Z}_{27}$	$\mathbb{Z}_3$	3

Ideas of proof

- Every minimal normal subgroup of A is solvable.
  By using that A_v is a $\{2,3\}$-group and K_3-simple group.

- Every normal abelian 3-subgroup M of A is isomorphic to $\mathbb{Z}_3$.
  By using quotient graphs relative to the orbits of M. If $M \ncong \mathbb{Z}_3$, then X is arc-transitive.

- $A \cong \mathbb{Z}_3 \times (\mathbb{Z}_p \rtimes \mathbb{Z}_9)$ with $9 \mid (p-1)$ or $\mathbb{Z}_p \rtimes \mathbb{Z}_{27}$ with $27 \mid (p-1)$.
  By using the edge-transitive graphs of order $3p$ or 9.

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Example (Definition 2.1, Praeger and Xu, European J. Combin., 1989)

Let $p \geq 3$. Then define the graph $C(3, p, 2) = (V, E)$ as follows:

$$V(X) = \mathbb{Z}_p \times \mathbb{Z}_3^2, \qquad E = \{\{(i, x, y), (i+1, y, z)\}\}$$

where $\mathbb{Z}_p$ and $\mathbb{Z}_3$ are additive groups of order p and 3, $i \in \mathbb{Z}_p$ and $x, y, z \in \mathbb{Z}_3$. Then $C(3, p, 2)$ is a connected hexavalent symmetric graphs of order $9p$ and $\mathrm{Aut}(C(3, p, 2)) = \mathrm{S}_3 \,\mathrm{wr}\, D_{2p}$.

Remark. It is easy to check that $C(3, 3p, 1) \cong C_{3p}[3K_1]$. Clearly, $C(3, p, 2)$ is not isomorphic to $C_{3p}[3K_1]$ because

$$\mathrm{Aut}(C(3, p, 2)) \neq \mathrm{Aut}(C_{3p}[3K_1]).$$

$\mathrm{Aut}(C(3, p, 2))$ has a minimal normal subgroup isomorphic to $\mathbb{Z}_3^p$, which is not semiregular on $V(C(3, p, 2))$.

Example (Definition 2.1, Praeger and Xu, European J. Combin., 1989)

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Examples: half-arc-transitive case

Let p be a prime and r an element of order 9 in $\mathbb{Z}_p^*$. Then $9 \mid (p-1)$. Suppose that $G(3p \rtimes 9) = \langle a, b, c \mid a^p = b^9 = c^3 = [a,c] = [b,c] = 1, b^{-1}ab = a^r \rangle \cong \mathbb{Z}_3 \times (\mathbb{Z}_p \rtimes \mathbb{Z}_9)$ with $r \neq 1$.

Construction 4.3, DM, 2019

Take $H = \langle b^3 c \rangle \leq G(3p \rtimes 9)$ and $g = ab$. Then $H \cong \mathbb{Z}_3$. Define the following coset graph:

$$\mathcal{HC}_{3p\rtimes 9}(9p) = \mathrm{Cos}(G(3p\rtimes 9), H, H\{g, g^{-1}\}H).$$

The coset graph $\mathcal{HC}_{3p\rtimes 9}(9p)$ is a connected hexavalent half-arc-transitive graph of order $9p$, and

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Construction 4.5, DM, 2019

Take $K = \langle y^9 \rangle \leq G(p{\rtimes}27)$. Then $K \cong \mathbb{Z}_3$. Set $g_1 = xy$, $g_2 = xy^2$, $g_3 = xy^4$. Define the following coset graphs:

$$\mathcal{HC}_{p\rtimes 27}(9p, 1) = \mathrm{Cos}(G(p{\rtimes}27), K, K\{g_1, g_1^{-1}\}K);$$

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The coset graphs $\mathcal{HC}_{p\rtimes 27}(9p, i)$ are connected hexavalent half-arc-transitive graphs of order $9p$, and for $i = 1, 2, 3$

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Thank you!