Addressing problem and the distance matrix of a graph

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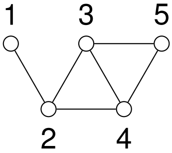
Outline

1. Addressing problem
2. Addressing and the determinant of the distance matrix
3. Other invariants on distance matrices which are constant among some classes of graphs.
4. Results and problems

Distance matrix

The distance matrix $D = D(G)$ of a graph G:

$D_{ij} = d_G(i, j)$ is the minimum length of a path from i to j



$$D = \begin{bmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 & 0 \end{bmatrix}$$

Addressing problem

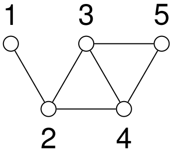
We now introduce a alphabet $\{0, 1, *\}$ and form addresses by taking *n*-tuples from this alphabet. The distance between two addresses is defined to be the number of places where one has a 0 and the other a 1 (so the stars do not contribute to the distance).

Example

$d(111**, 10*1*) = 1$, $d(10*1*, *0001) = 1$,
$d(10101, *****) = 0$, $d(111**, 00000) = 3$.

For an addressing of a graph G, we require that the distance of any two vertices in G is equal to the distance of their addresses.

Example



1	1	1	1	*	*
2	1	0	*	1	*
3	*	0	0	0	1
4	0	0	1	*	*
5	0	0	0	0	0

and

1	0	1	1	1
2	0	1	*	0
3	1	*	0	*
4	0	0	*	0
5	1	0	1	0

$N(G)$ and the lower bound

$N(G)$: the minimum value of k for which there exists an addressing of G with length k.

It is known that finding an addressing of length k is equivalent to finding 0-1 matrices X, Y with k columns such that

$$XY^T + YX^T = D(G).$$

Let n_+, respectively n_-, be the number of positive, respectively negative, eigenvalues of the distance matrix $D(G)$ of the graph G.

Theorem (Graham and Pollak, 1971)

$$N(G) \geq \max\{n_+, n_-\}.$$

$\square$

If an addressing of G is of length $\max\{n_+, n_-\}$, we say the addressing is eigensharp.

Upper bound of $N(G)$

In 1971, Graham and Pollak conjectured that

$$N(G) \leq |V(G)| - 1.$$

This is proved by Winkler in 1983.

Theorem (Winkler, 1983)

$N(G) \leq |V(G)| - 1.$ □

History

- For tree T, $N(T) = n(T) - 1$; $N(C_{2n+1}) = 2n$, $N(C_{2n}) = n$. [Graham and Pollak, 1971]
- For a block graph G, $N(G) = n - 1$. (A consequence of some known result)
- For Petersen graph P, $N(P) = 6$ and it has no eigensharp adressing ($n_- = 5$). [Elzinga, Gregory and Vander Meulen, 2004]
- The number $N(K_{m,n})$ for complete bipartite graphs. [Fujii and Sawa, 2008]
- $N(G) = \sum_{i=1}^{k}(n_i - 1)$ for $G = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_k}$. [Cioaba, Elzinga et al, 2018]
- $N(J(n,k)) \leq k(n-k)$ for Johnson graphs. [Alon, Cioaba, Gilbert, Koolen and McKay, 2019]

Inertia and principal leading minors

The inertia of a matrix is (n_+, n_-, n_0).

Lemma (Jones, 1950)

Let A be a nonsingular symmetric $n \times n$ matrix with principal leading minors $D_1, \ldots, D_n$. If there is no consecutive two zeros in the sequence $D_1, \ldots, D_n$, then n_- is the number of sign changes in the sequence $1, D_1, \ldots, D_n$, ignoring the zeros in the sequence. $\square$

Example

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix},$$

$$(1, D_1, D_2, D_3, D_4) = (1, -2, 3, -4, 5), \quad \mathrm{inertia}(A) = (0, 4, 0).$$

Examples

Theorem (Graham and Pollak, 1971)
For tree T of order n, $\det(D(T)) = (-1)^{n-1}(n-1)2^{n-2}$ $\square$

Let T be a tree of order n. From the theorem above, we have

$$(1, D_1, \ldots, D_n) = (+, 0, -, +, -, \ldots),$$

so $\mathrm{inertia}(D(T)) = (1, n-1, 0)$ and $N(T) = n - 1$.

Since $D(K_n) = A(K_n)$, $\det(D(K_n)) = (-1)^{n-1}(n-1)$. We also have $\mathrm{inertia}(D(K_n)) = (1, n-1, 0)$ and $N(K_n) = n - 1$.

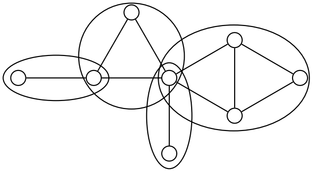
$\operatorname{cof}(A)$

The (i, j)-th cofactor of A is $(-1)^{i+j}\,\mathtt{det}(A(i|j))$, where $A(i|j)$ is the matrix obtained from A by deleting the i-th row and the j-th column.

Define $\operatorname{cof}(A)$ be the sum of all cofactors of A.

Blocks of a graph

In a graph, a *block* is a maximal subgraph without a cut vertex.



General case

Theorem (Graham, Hoffman and Hosoya, 1977)

Let G be a connected graph with $V(G) = 1, \ldots, n$. Let $G_1, \ldots, G_k$ be the blocks of G. Then the following assertions hold:

(i) $\operatorname{cof}(D(G)) = \prod_{i=1}^{k} \operatorname{cof}(D(G_i))$

(ii) $\det(D(G)) = \sum_{i=1}^{k} \det(D_{G_i}) \prod_{j \neq i}^{k} \operatorname{cof}(D(G_j))$.

□

Theorem

If each block of G of order n is a complete graph, then $N(G) = n - 1$.

Smith normal form

Given an $m \times n$ integer matrix P, there exist matrices U, V which are invertible in $M_m(\mathbb{Z})$ and $M_n(\mathbb{Z})$ respectively such that $UPV = S$ and

(i) The off-diagonal entries of S are all zero.

(ii) The diagonal entries $s_1, s_2, \ldots$ of S satisfies $s_i | s_j$ for $i < j$.

The matrix S (which is unique) is called the *Smith normal form* of P, denoted by $\mathtt{Snf}(P)$.

Smith normal form

In 2008, Hou and Woo showed that

- For tree T of n vertices, $\mathrm{Snf}(D(T)) = I_2 \oplus 2_{n-3} \oplus (2n-2)$.
- $\mathrm{Snf}(D(C_n)) = \begin{cases} I_{n-1} \oplus (\frac{n^2-1}{4}), & \text{if } n \text{ is odd} \\ I_2 \oplus 2I_{\frac{n}{2}-2} \oplus (n) \oplus 0I_{\frac{n}{2}-1}, & \text{if } n \text{ is even} \end{cases}$.

In 2016, Chen and Hou showed that for block graphs G of order n with blocks $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$,

$$\mathrm{Snf}(D(G)) = I_{n-k-1} \oplus \begin{bmatrix} k & 1 & 1 & \cdots & 1 & 1 \\ 1 & n_1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & n_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & n_{k-1} & 0 \\ 1 & 0 & 0 & \cdots & 0 & n_k \end{bmatrix}$$

Classes of graphs with constant N, det, cof, $\mathsf{inertia}$, Snf

In 2013, Chang and Gosil showed that $\mathsf{inertia}(D(G))$ depends only on the blocks of G. In their proof, it is also garanteed that $\mathsf{Snf}(D(G))$ also depend only on the blocks of G.

Therefore, if the blocks of G are given, each of $\mathsf{det}(D(G))$, $\mathsf{cof}(D(G))$, $\mathsf{inertia}(D(G))$, $\mathsf{Snf}(D(G))$ is constant. Furthermore, if each block is a clique, $N(G)$ is constant.

Do other classes of graphs have constant $\mathsf{det}(D(G))$, $\mathsf{cof}(D(G))$, $\mathsf{inertia}(D(G))$, $\mathsf{Snf}(D(G))$ or $N(G)$, or some other constant invariants?

The 2-clique paths

Let $p_1, \ldots, p_m$ be integers at least 3. A 2-clique path is obtained by gluing an edge of K_{p_i} to an edge of $K_{p_{i+1}}$, $i = 1, \ldots, m$; an edge cannot be glued twice. The set of such graphs is denoted by $\mathcal{CP}_{p_1, p_2, \ldots, p_m}$.

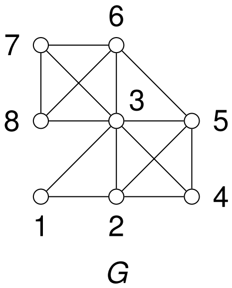


Figure: A graph G in $\mathcal{CP}_{2:3,4,3,4}$

Theorem

Let $G \in \mathcal{CP}_{2:p_1,\dots,p_m}$ and $n = |V(G)|$. Then

$$\det(D(G)) = (-1)^{n-1} \left(1 + \sum_{k \text{ odd}} (p_k - 2)\right) \left(1 + \sum_{k \text{ even}} (p_k - 2)\right),$$

$\operatorname{inertia}(D(G)) = (1, n-1, 0)$ *and* $N(G) = n - 1$. □

Theorem

Let $G \in \mathcal{CP}_{2:p_1,\dots,p_m}$ and $n = |V(G)|$. Then

$$\operatorname{cof}(D(G)) = (-1)^{n-1} n.$$

Corollary

Let G be a linear 2-tree on n vertices. Then

$$\det(D(G)) = (-1)^{n-1} \left(1 + \left\lfloor \frac{n-2}{2} \right\rfloor\right) \left(1 + \left\lceil \frac{n-2}{2} \right\rceil\right)$$

and $\operatorname{cof}(D(G)) = (-1)^{n-1} n$. □

Applications on addressing problem

Theorem

If G is a connected graph of order n whose blocks are 2-clique paths, then

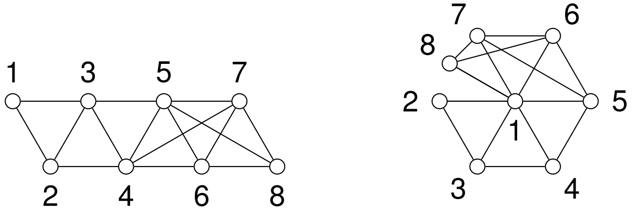
$$\mathrm{inertia}(D(G)) = (1, n-1, 0)$$

and $N(G) = n - 1$. $\square$

CP (clique-path) graphs $\mathcal{CP}_s$

Let $s = q_1, q_2, \ldots, q_n$. ($q_i \geq 2$ for $i \geq 3$.)

- vertex k and its backward neighbors form a clique
- $|N(k) \cap [k-1]| = q_k$
- $N(k) \cap [k-1] = \{a_k\} \cup [b_k, k-1]$, where $b_k = k - q_k + 1$ is fixed and a_k may vary.



G_1 G_2

Figure: Two graphs G_1 and G_2 in $\mathcal{CP}_{0,1,2,2,2,2,3,3}$

a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1	2	3	4	4	5

a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1	1	1	1	1	1

CP (clique-path) graphs $\mathcal{CP}_s$

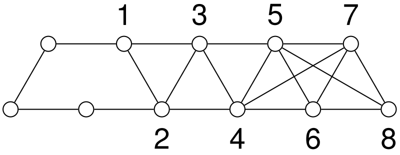
Theorem (Cheng and Lin, 2018)

Let s be a sequence. For any $G \in \mathcal{CP}_s$, the matrix $E^T D(G) E$ is only depend on s, where E is an upper triangular matrix with 1 on the diagonal. $\square$

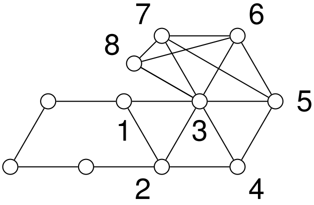
Corollary

Let s be a sequence. Then for any graph $G \in \mathcal{CP}_s$, $\operatorname{det}(D(G))$, $\operatorname{cof}(D(G))$, $\operatorname{inertia}(D(G))$ and the Smith normal form of $D(G)$ are uniquely determined by s. $\square$

Attaching the CP graphs



G_1



G_2

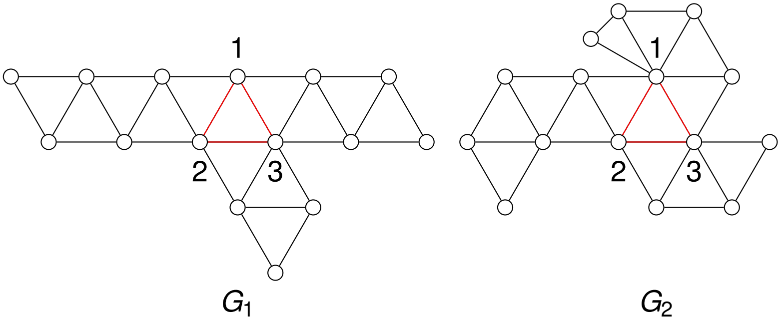
$$\det(D(G_1)) = \det(D(G_2)).$$

$$\mathrm{cof}(D(G_1)) = \mathrm{cof}(D(G_2)).$$

$$\mathrm{inertia}(D(G_1)) = \mathrm{inertia}(D(G_2)).$$

$$\mathrm{Snf}(D(G_1)) = \mathrm{Snf}(D(G_2)).$$

Attaching the CP graphs

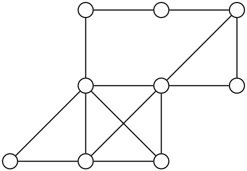


$$\det(D(G_1)) = \det(D(G_2)) = 56.$$

2-cycle-clique path

Let $(G_1, G_2, \ldots, G_k)$ be a sequence of graphs with G_i a cycle or a clique for $1 \leq i \leq k$.

A 2-cycle-clique path is obtained by gluing an edge of G_i to an edge of G_{i+1}, $i = 1, \ldots, m$; an edge cannot be glued twice.



(K_3, K_4, C_5, K_3)

2-cycle-clique path

Theorem (Cheng and Lin, 2019+)

Let G be a 2-cycle-clique path corresponding to $G_1, G_2, \ldots, G_k$ of order n. Then $\det(G)$ is a constant.

Concluding remark

Classes of graphs with constant invariants (on distance matrix)

- Determined by blocks: $\mathsf{Snf}, \mathsf{inertia}, \det, \mathsf{cof}$; N when every block is a clique
- CP graphs, 2-clique path: $\mathsf{Snf}, \mathsf{inertia}, \det, \mathsf{cof}$
- 2-cycle-clique path: $\det, \mathsf{cof}$
- A graph attach CP graphs: $\mathsf{Snf}, \mathsf{inertia}, \det, \mathsf{cof}$

Problem: Do other classes of graphs have constant some invariants?
Problem: Can we describe $\mathsf{inertia}(G)$ or $\mathsf{Snf}(G)$ by the blocks of G?
Problem: Are there some relations between $\mathsf{Snf}, \mathsf{inertia}, \det, \mathsf{cof}$?

Thank you for your attention

Reference I

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