On Some Gallai Ramsey Numbers

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August, 2019

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1. Introduction

Gallai Ramsey number

A *Gallai k-coloring* is
a k-edge-coloring of a complete graph K_N such that
no triangle has all its edges colored differently.

Given a graph H and an integer $k \geq 1$,
the *Gallai Ramsey number* $GR_k(H)$ of H is
the least positive integer N such that
every Gallai k-coloring of the complete graph K_N
contains a monochromatic copy of H.

Original definition of k-color Ramsey number

Let G_i be a simple graph of order n_i, $1 \le i \le k$.

The *Ramsey number* $R(G_1, G_2, \ldots, G_k)$ is the minimum integer N with the following property:

If the edges of K_N are colored by k colors, then there exists some i with $1 \le i \le k$ such that K_N has a subgraph in color i, which is isomorphic to G_i.

If $G_1 = G_2 = \cdots = G_k = H$, we just write

$$R(G_1, G_2, \ldots, G_k) = R_k(H).$$

Background of Gallai coloring

- ☞ T. Gallai, Transitiv orientierbare Graphen, *Acta Math. Acad. Sci. Hung.* 18(1967) 25–66.
- ☞ Information theory: entropy of graphs
- ☞ Perfect graph
- ☞ Partially ordered sets

Let $R_k(H)$ be the k-color classical Ramsey number for H, then it is easy to see that

$$GR_k(H) \leq R_k(H) \text{ for any } graph\ H.$$

Theorem 1. For an integer $k \geq 1$ and a graph H with no isolated vertices, $GR_k(H)$ is *exponential* in k if H is not bipartite, *linear* in k if H is bipartite but not a star, and *constant* (does not depend on k) when H is a star.

[Gyárfás et al., JGT, 64(2010), 233-243.]

2. Cycles

If H is a cycle, the by Theorem 1, $GR_k(C_{2n})$ is *linear* in k, and $GR_k(C_{2n+1})$ is *exponential* in k.

Theorem 2. For all $k \geq 1$ and $n \geq 3$,

$$(n-1)k+n+1 \leq GR_k(C_{2n}) \leq (n-1)k+3n.$$

[Hall et al., JGT, 75(2014), 59-74]

Theorem 3. For all $k \geq 1$ and $n \geq 2$,

$$n \cdot 2^k+1 \leq GR_k(C_{2n+1}) \leq (n \ln n) \cdot (2^{k+3}-3).$$

[Hall et al., JGT, 75(2014), 59-74]

Theorem 4. For all $k \geq 2$ and $n \geq 2$,

$$n \cdot 2^k + 1 \leq GR_k(C_{2n+1}) \leq (4n + n\ln n) \cdot 2^k.$$

[Chen et al., submitted]

Theorem 5. For all $k \geq 3$ and $n \geq 8$,

$$n \cdot 2^k + 1 \leq GR_k(C_{2n+1}) \leq (n\ln n) \cdot 2^k - (k+1)n + 1.$$

[Bosse et al., submitted]

Except these general bounds for cycles, some exact values of $GR_k(C_{2n})$ and $GR_k(C_{2n+1})$ are determined for n is small.

Theorem 6. $GR_k(C_5)=2\cdot 2^k+1$.

[Fujita and Magnant, DM, 311(2011) 1247-1254]

Theorem 7. $GR_k(C_7)=3\cdot 2^k+1$.

[Bruce and Song, DM, 342(2019) 1191-1194]

Theorem 8. $GR_k(C_9)=4\cdot 2^k+1$ and $GR_k(C_{11})=5\cdot 2^k+1$.

[Bosse and Song, submitted]

Theorem 9. $GR_k(C_{10})=4k+6$ and $GR_k(C_{12})=5k+7$.

[Lei et al., submitted]

Our Results

Let C_m denote a cycle of length m.

Our main results are as follows.

Theorem 10. For all $k \geq 1$ and $n \geq 3$,

$$GR_k(C_{2n}) = (n-1)k + n + 1.$$

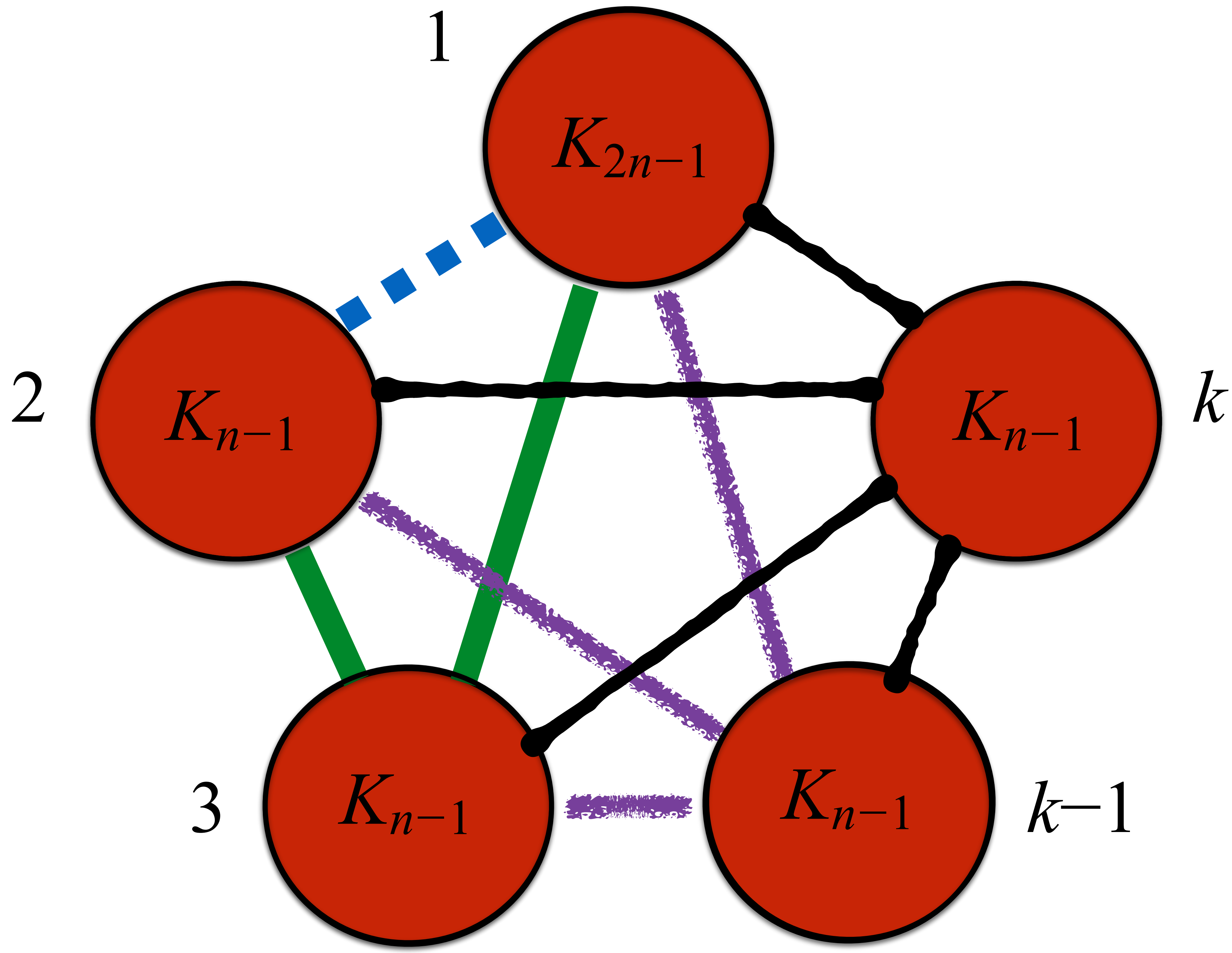
[Zhang et al., preprint, 2018]

Theorem 11. For all $k \geq 1$ and $n \geq 2$,

$$GR_k(C_{2n+1}) = n \cdot 2^k + 1.$$

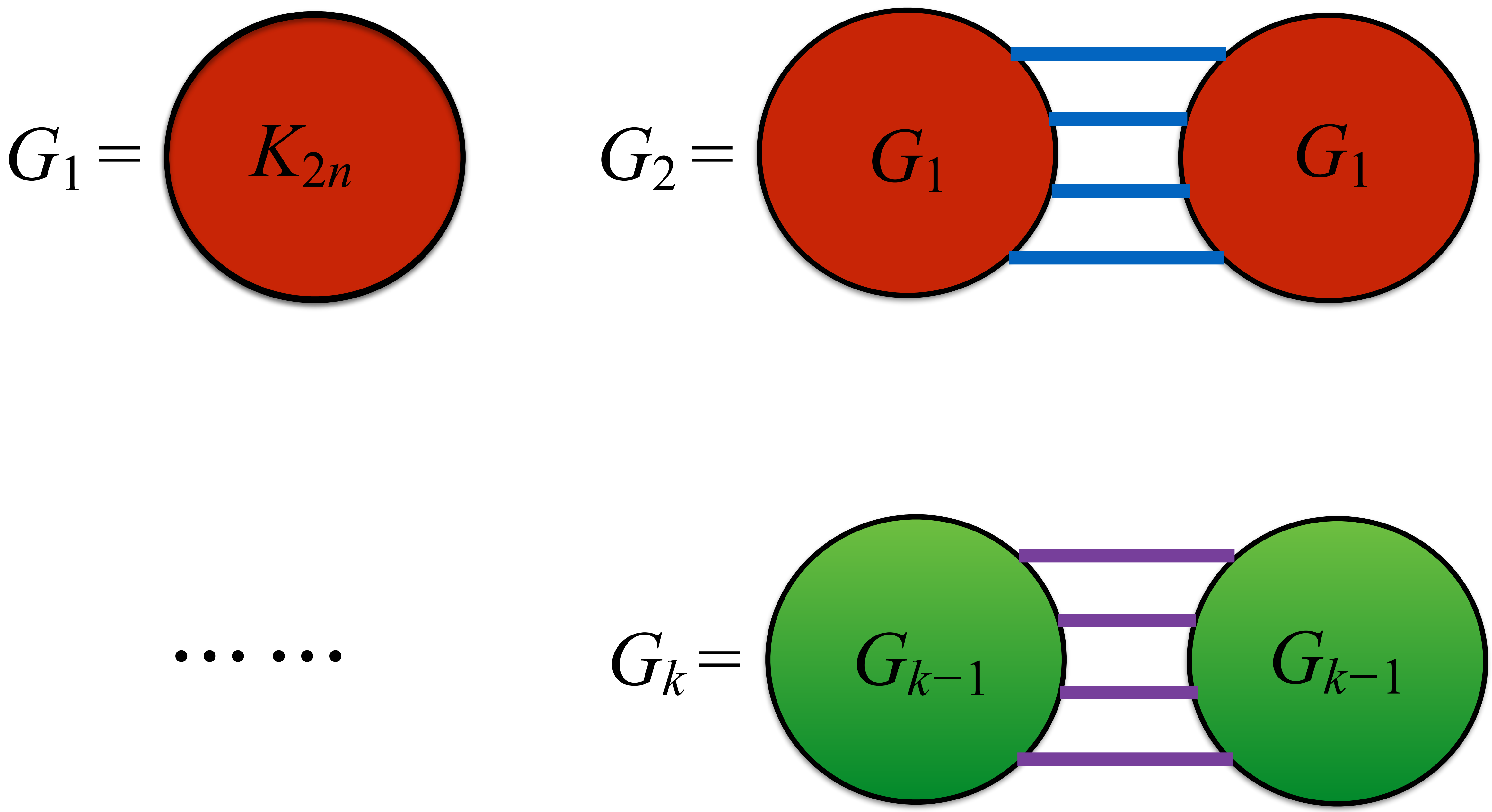
[Zhang et al., preprint, 2018]

Low bound for $GR_k(C_{2n})$, where $k \geq 1$ and $n \geq 3$:
$GR_k(C_{2n}) \geq (n-1)k+n+1$.



Low bound for $GR_k(C_{2n+1})$, where $k \geq 1$ and $n \geq 3$:

$GR_k(C_{2n+1}) \geq n \cdot 2^k + 1$.

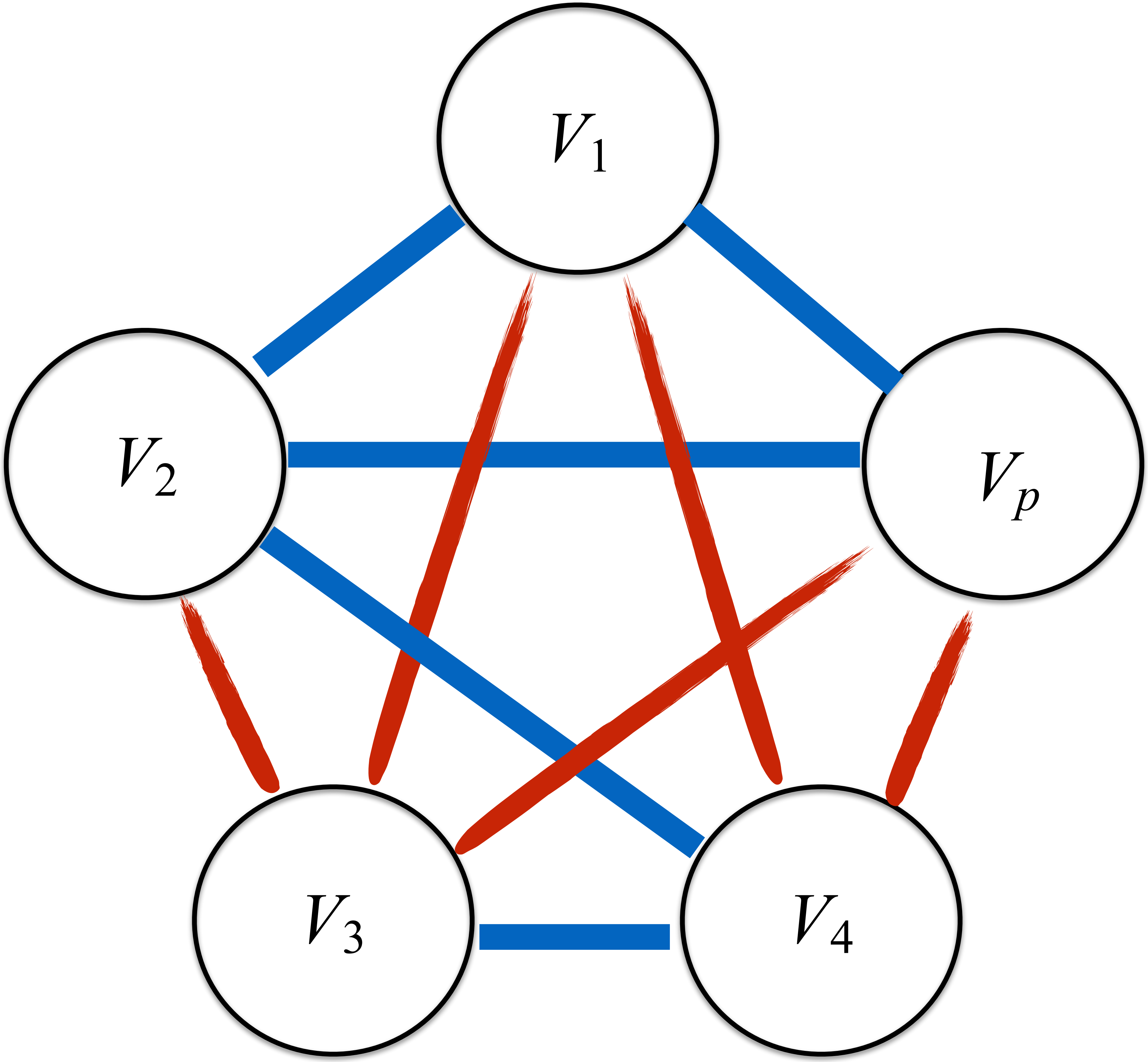


An important structural result of Gallai on Gallai colorings of complete graphs:

Theorem A. For any Gallai coloring c of a complete graph G with $|G| \geq 2$, $V(G)$ can be partitioned into nonempty sets $V_1, V_2, \ldots, V_p$ with $p > 1$ so that at most two colors are used on the edges in

$$E(G) \setminus (E(V_1) \cup \cdots \cup E(V_p))$$

and only one color is used on the edges between any fixed pair (V_i, V_j) under c, where $E(V_i)$ denotes the set of edges in $G[V_i]$ for all i, $1 \leq i \leq p$.



V_1
V_2
V_p
V_3
V_4

Sketch of the Proofs of Theorems 10 and 11

For any Gallai k-colored complete graph G with $|G|\geq n\geq 2$ and color classes $E_1,\ldots,E_k$, let $q(G)$ denote the number of colors $i\in\{1,2,\ldots,k\}$ such that H_i with $V(H_i)=V(G)$ and $E(H_i)=E_i$ has a component of order at least n.

Theorem 12. Let G be a Gallai k-colored complete graph with $|G|\geq n\geq 3$. If $|G|\geq(n-1)\cdot q(G)+n+1$, then G has a monochromatic C_{2n}.

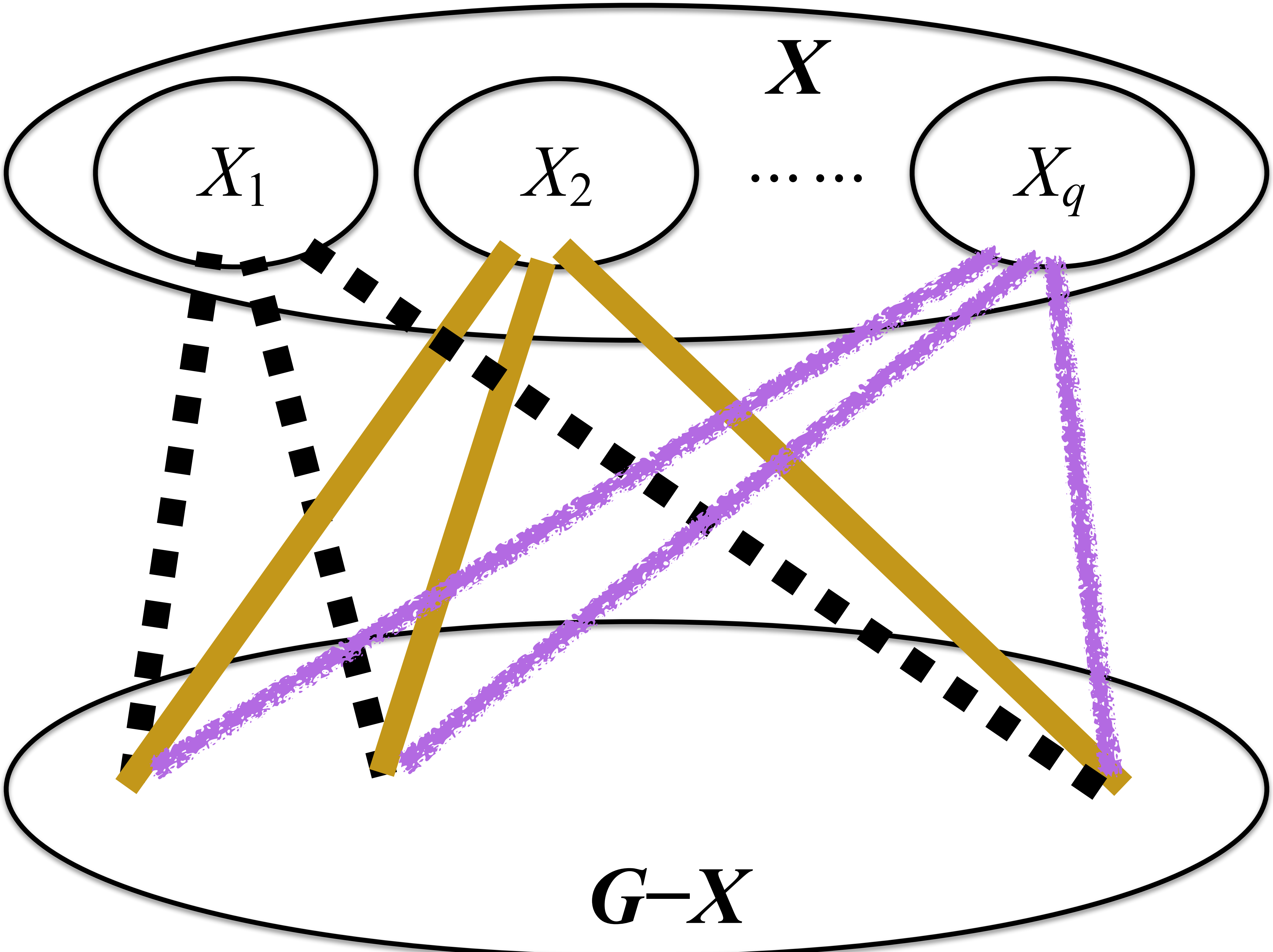
Suppose G has no monochromatic copy of C_{2n}. Choose G with $q=q(G)$ minimum. Assume that for each color $i \in \{1,2,\ldots,q\}$, H_i has a component of order at least n.

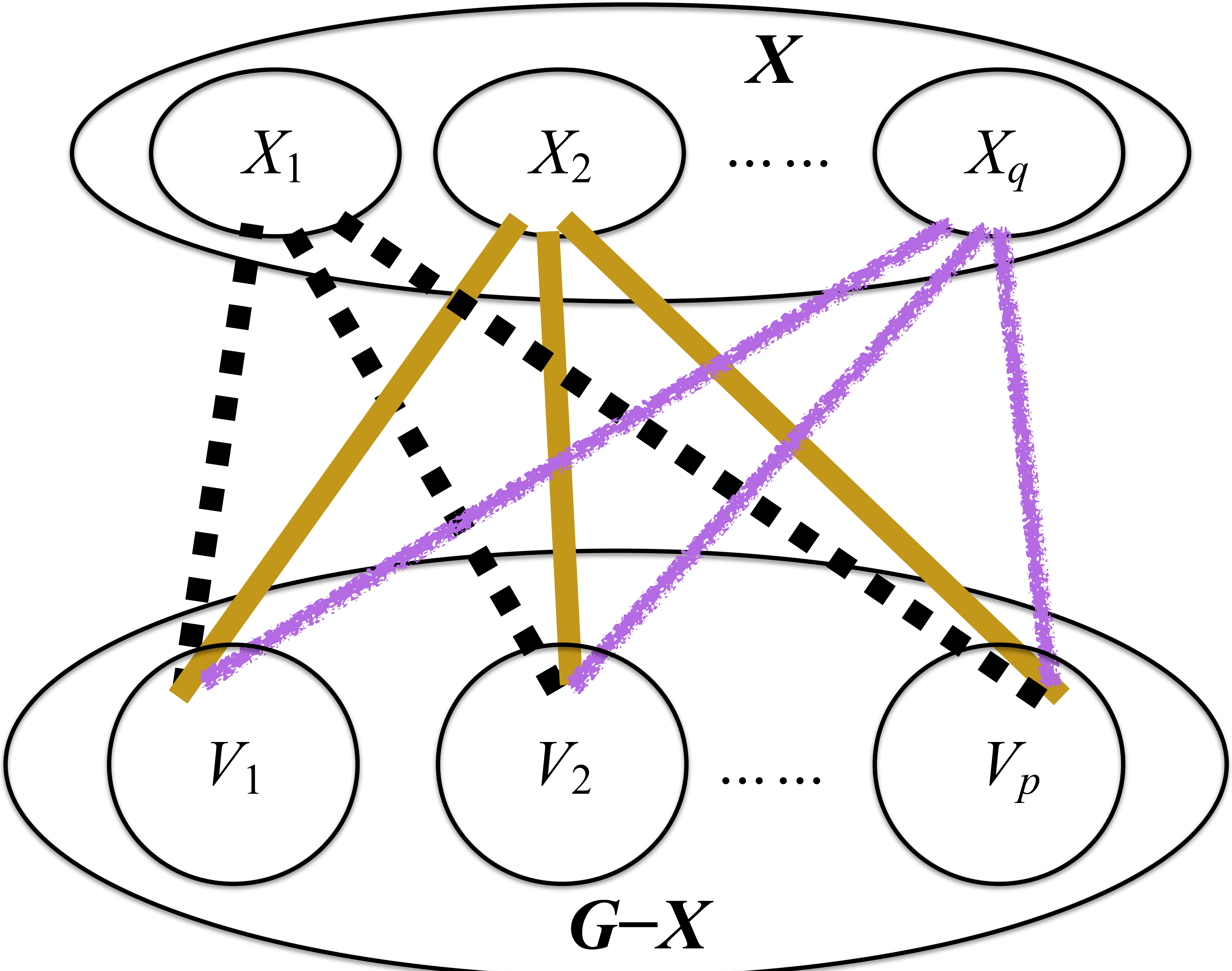
Let $X_1,\ldots, X_q$ be disjoint subsets of $V(G)$ such that for each $i \in \{1,2,\ldots,q\}$, X_i (*possibly empty*) is mc-complete in color i to $V(G)\backslash(X_1 \cup \cdots \cup X_q)$.

Let $X=X_1 \cup \cdots \cup X_q$.

Choose $X_1,\ldots, X_q$ such that

$|X|$ is as large as possible subject to $|X|\leq|G|-n$.


X
X_1
X_2
……
X_q
G−X


X
X_1
X_2
……
X_q
V_1
V_2
……
V_p
G−X

3. Complete Bipartite Graphs

Let $K_{m,n}(m \geq n)$ be a complete bipartite graphs. By Theorem 1, $GR_k(K_{m,n})$ is *linear* in k.

It is known that

$$GR_k(K_{m,n}) \geq (n-1)(k-2) + R_2(K_{m,n}).$$

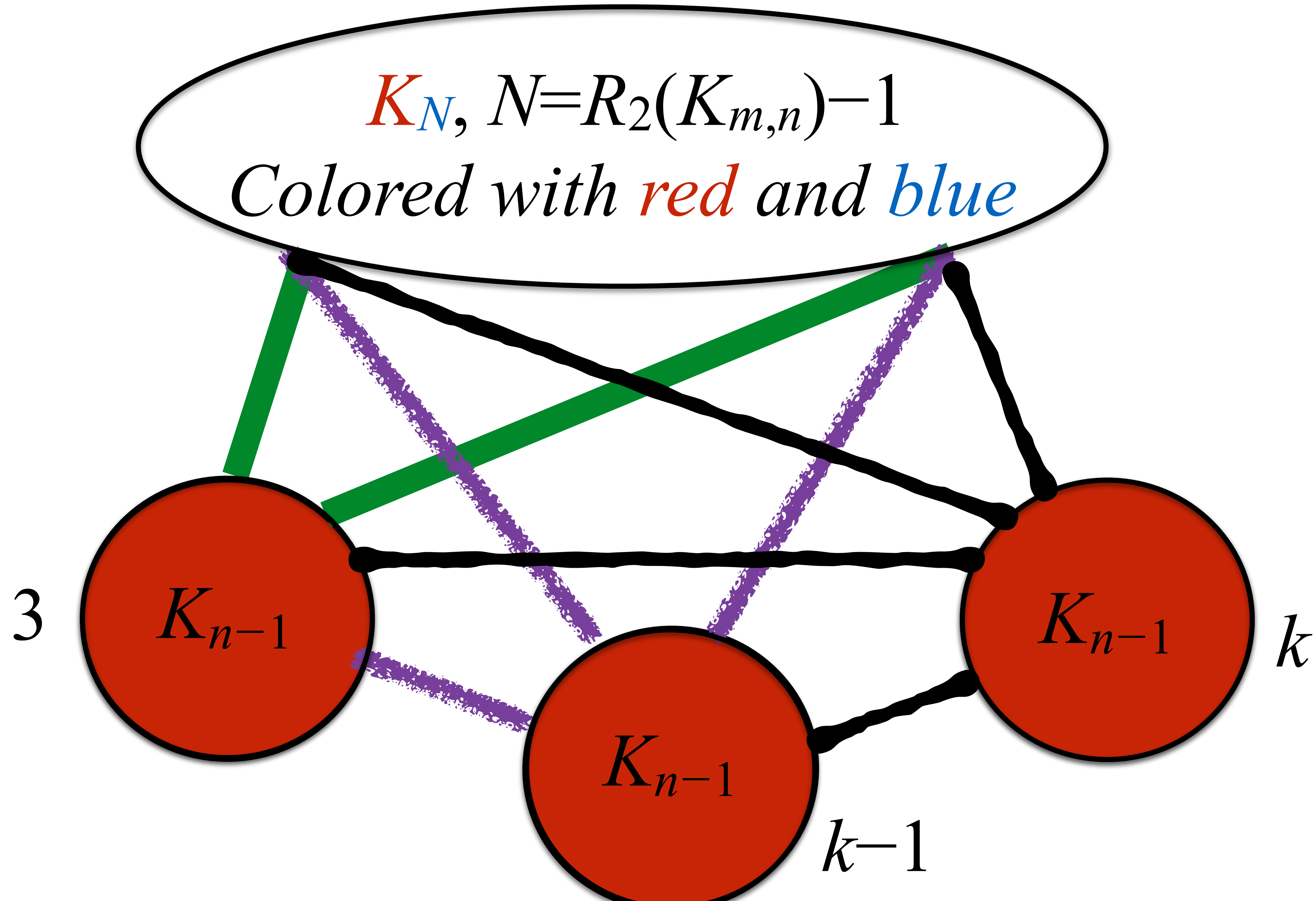
Conjecture 1. Let $k \geq 2$ and $m \geq n \geq 1$ be integers. If $R_2(K_{m,n}) \geq 3m-2$, then

$$GR_k(K_{m,n}) = (n-1)(k-2) + R_2(K_{m,n}).$$

[Wu et al., DAM, 254(2019),196-203.]

Lower bound for $GR_k(K_{m,n})$:

$$GR_k(K_{m,n}) \geq (n-1)(k-2) + R_2(K_{m,n})$$



Theorem 13. For integers $k \geq 2$, $m \geq 2$ and $n \geq 2$,

$$GR_k(K_{m,n}) \leq [R_2(K_{m,n})-1][(m+n-1)^2 k+2(m+n+1)].$$

[Gyárfás et al., JGT, 64(2010), 233-243.]

Theorem 14. For fixed integers $k \geq 2$ and $m \geq 1$, if $n \rightarrow \infty$, then

$$GR_k(K_{m,n}) \leq (2^n+2^{n/2+1}+k)m+4n^3.$$

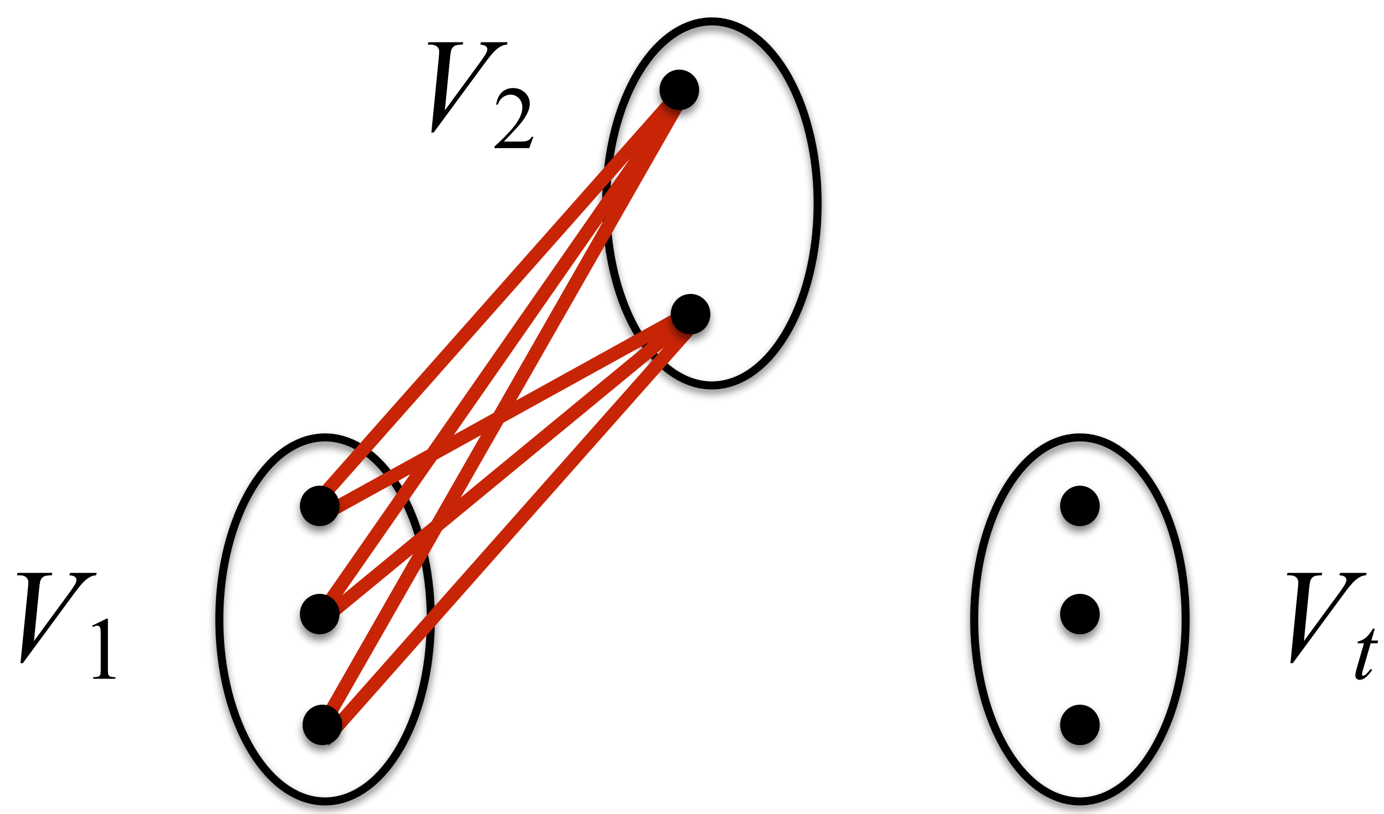
[Chen et al., G&C, 34(2018), 1185-1196.]

Theorem 15. Let $R=max\{R_2(K_{m,n}),3m-2\}$. Then for integers $k \geq 2$ and $m \geq n \geq 1$,

$$GR_k(K_{m,n}) \leq (n-1)(k-3)+(n-1)R+1.$$

[Wu et al., DAM, 254(2019),196-203.]

A *uniform k-coloring* of
a complete multipartite graph $G=(V_1,\ldots,V_t)$
is a *k-edge coloring* such that
the edges between any two parts
receive the same color.



Let $l \geq 1$ and $G_1,\ldots,G_k$ be k graphs.
The *l-uniform Ramsey number* $R^l(G_1,\ldots,G_k)$ is defined as the minimum integer N such that, any uniform k-coloring of a complete multipartite graph on N vertices with each part of cardinality no more than l, must contain a monochromatic copy G_i in color i for some i.

$R^l(G_1,\ldots,G_k)$ is well-defined because

$$R^l(G_1,\ldots,G_k) \leq l \cdot (R(G_1,\ldots,G_k) - 1) + 1.$$

$$R^1(G_1,\ldots,G_k) = R(G_1,\ldots,G_k).$$

In general,

$$R(G_1,\ldots,G_k) \leq R^l(G_1,\ldots,G_k) \leq l\cdot(R(G_1,\ldots,G_k)-1)+1.$$

Remark 1. If each G_i is a complete graph, then

$$R^l(G_1,\ldots,G_k) = l\cdot(R(G_1,\ldots,G_k)-1)+1.$$

If $G_1=\cdots=G_k=H$, we write

$$R_k^\ell(H) = R^\ell(G_1, \ldots, G_k)$$

Theorem 16. Let $k\geq 3$ and $m\geq n\geq 2$ be integers. Then

$$GR_k(K_{m,n}) = (n-1)(k-2) + R_2^{m-1}(K_{m,n}).$$

[Liu and Chen, Preprint, 2019.]

Theorem 17. For $k \geq 3$ and $m \geq n \geq 2$,

$$R_2^{m-1}(K_{m,n}) \leq 2R_2(K_{m,n}) + n - 2.$$

[Liu and Chen, Preprint, 2019.]

Theorem 18. Let $m \geq 4$. If $R_2(K_{3,m}) \geq 11m/2-4$, then

$$R_2^{m-1}(K_{3,m}) = R_2(K_{3,m}).$$

[Liu and Chen, Preprint, 2019.]

Theorem 19. For $k \geq 2$,

$$GR_k(K_{3,4})=2(k-2)+R_2(K_{3,4}).$$

[Liu and Chen, Preprint, 2019.]

4. Complete Graphs

For H being a complete graph K_t,

Fox and Grinshpun posed the following.

Conjecture 1. For $k \geq 1$ and $t \geq 3$,

$$GR_k(K_t) = \begin{cases} [R_2(K_t)-1]^{k/2}+1 & \text{if } k \text{ is even,} \\ (t-1)[R_2(K_t)-1]^{(k-1)/2}+1 & \text{if } k \text{ is odd.} \end{cases}$$

[Fox and Grinshpun, JCTB, 111(2015), 75-125.]

Conjecture 1 is true for $t=3$.

[Chung and Graham, Comb., 3(1983), 315-324.]

Thank you for your attention!