IS EVERY PRIME SUM GRAPH HAMILTONIAN?

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outline

WHY

WE STUDY

HOW

WE DID

WHAT

RELEVANT

BERTRAND'S POSTULATE

(Bertrand-Chebyshev Thm.)

For any positive integer $n>1$ there exists a prime number p between n and $2n$

CONSEQUENCE OF BERTRAND'S POSTULATE

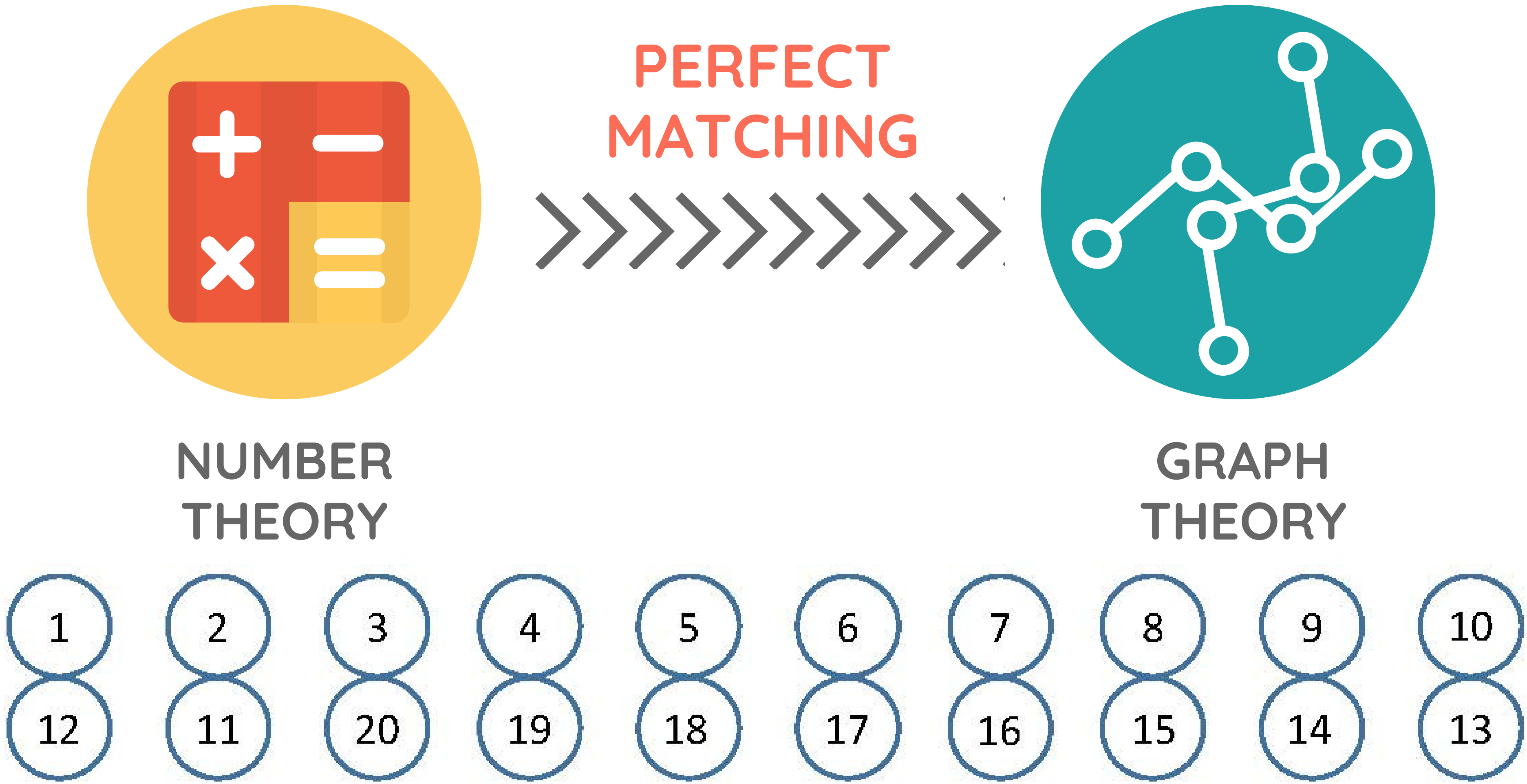
1998 proved by
L. Greenfield & S. Greenfield

2006 reproduced by
D. Galvin

"For any positive integer n, { 1, 2, ..., 2n } can be paired such that the sum of each pair is a prime."

- D. Galvin, Erdos's proof of Bertrand's postulate, April 2006.
- L. Greenfield and S. Greenfield, Some problems of combinatorial number theory related to Bertrand's postulate, J. Integer Seq. 1 (1998), Article 98.1.2.

NEW INSIGHTS



PRIME SUM GRAPH

For any positive number n,
define $G_n = (V, E)$ with
$V=\{1, 2, ..., n\}$ and
$E=\{ ij : i+j \text{ is prime}\}$

Greenfield & Greenfield
"G_{2n} has a perfect matching."

HUNG-LIN FU

HONG-BIN CHEN

JUN-YI GUO

CONJECTURE

"G_{2n} has a Hamilton cycle"

"that is,
for any $2n>2$, $\{ 1, 2, ..., 2n \}$ can
be rearranged into a **cycle**
so that the **sum** of every
two **adjacent numbers** is **prime**"

Douglas B. West's page

Traversal by Prime Sum

Originator(s): ????

Question: Let G_m be the graph with vertex set $\{1,2,\ldots,2m\}$ such that xy is an edge if and only if $x+y$ is prime. Is G_m Hamiltonian when $m >= 2$?

Comments/Partial results: It is easy to build a Hamiltonian cycle when $2m+1$ and $2m+3$ are both prime, but it is not even known if G_m is Hamiltonian for infinitely many m.

References: This question was discussed in a thread on the now-defunct mailing list COMB-L.

Index Page; *Glossary*

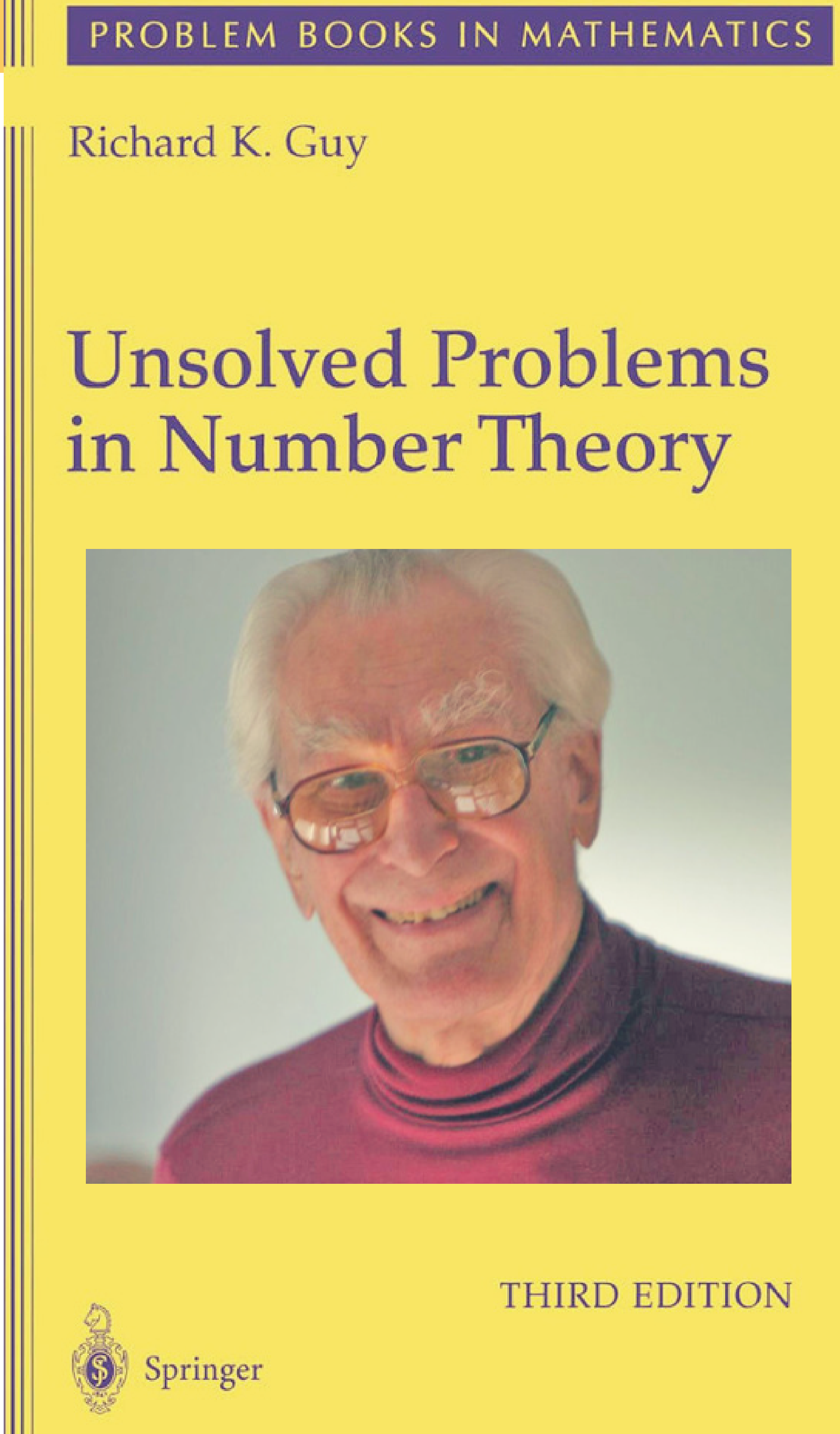
Posted 7/11/03

quote from p.105-106

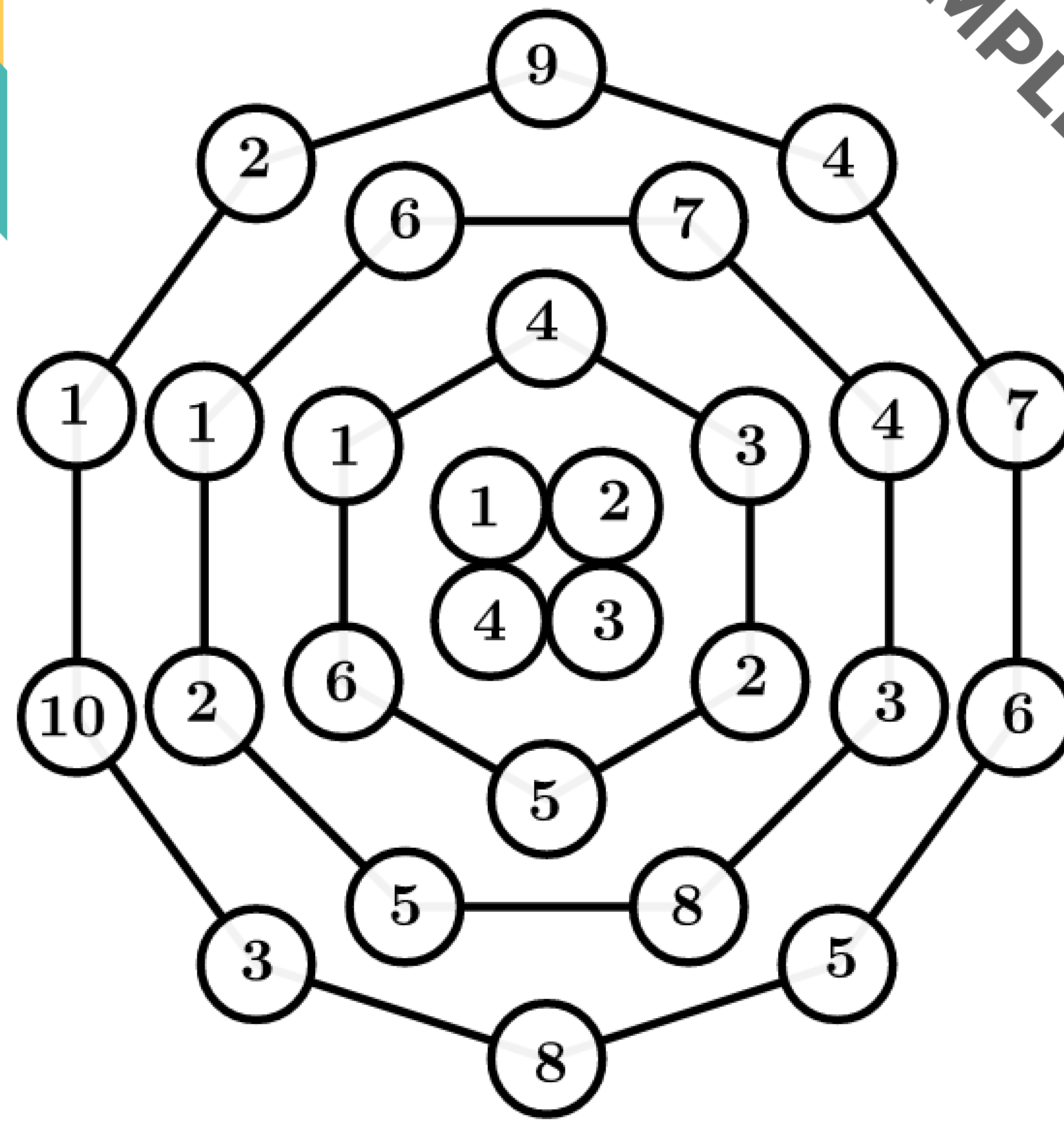
"Antonio Filz (1982) defined a prime circle of order 2m to be a circular permutation of numbers from 1 to 2m with each adjacent pair summing to a prime.

There is essentially only one prime circle for m=1, 2, and 3; two for m=4 and 48 for m=5.

Are there prime circles for all m? "



EXAMPLES



sum of adjacent numbers = prime

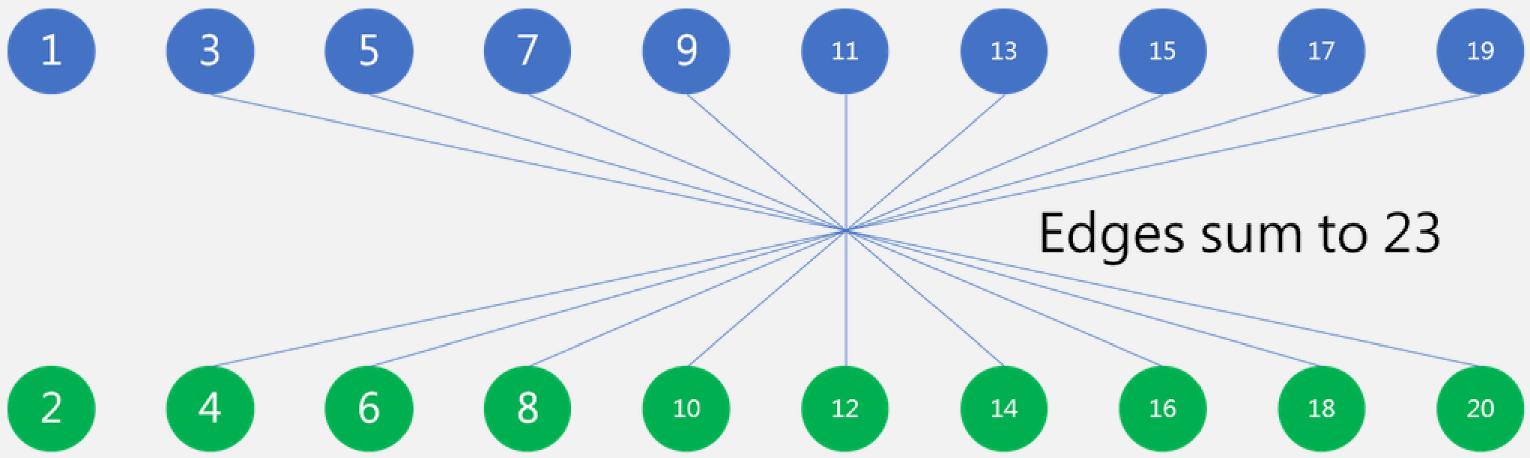

HOW WE DID

Observation

Bipartite graph

Symmetry

……



SUFFICIENT CONDITION

THEOREM [CFG 2018]

IF { p, q, $2n+p$, $2n+q$ } are primes
$\gcd((q-p)/2, n) = 1$ (p can be 1)

THEN G_{2n} has a Hamilton cycle

4 primes

1 gcd condition

01

{ 1, 3, 2n+1, 2n+3 }

gcd condition holds directly

at least 3 primes are required

02

If twin prime conjecture is true

then Filz's conjecture is verified for **infinitely many cases**

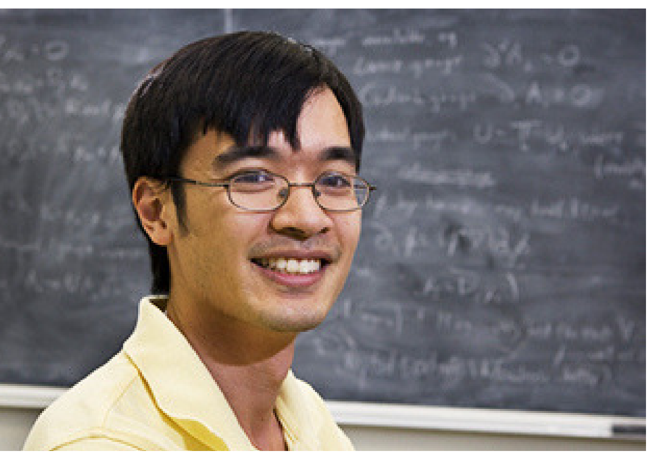
03

DIFFICULTY

Need to prove there are infinitely many prime triples (or quadruples) satisfying certain conditions

BREAKTHROUGH OF TWIN PRIME CONJECTURE

246 **600** **70M**



Yitang Zhang showed in 2013 that we will never stop finding pairs of primes that are within a bounded distance — within 70 million. Soon after, dozens of outstanding researchers in the world bring it down to 246.

POLYMATH PROJECT | **JAMES MAYNARD** | **YITANG ZHANG**

THANKS TO BREAKTHROUGH

OUR SUFFICIENT CONDITION

+

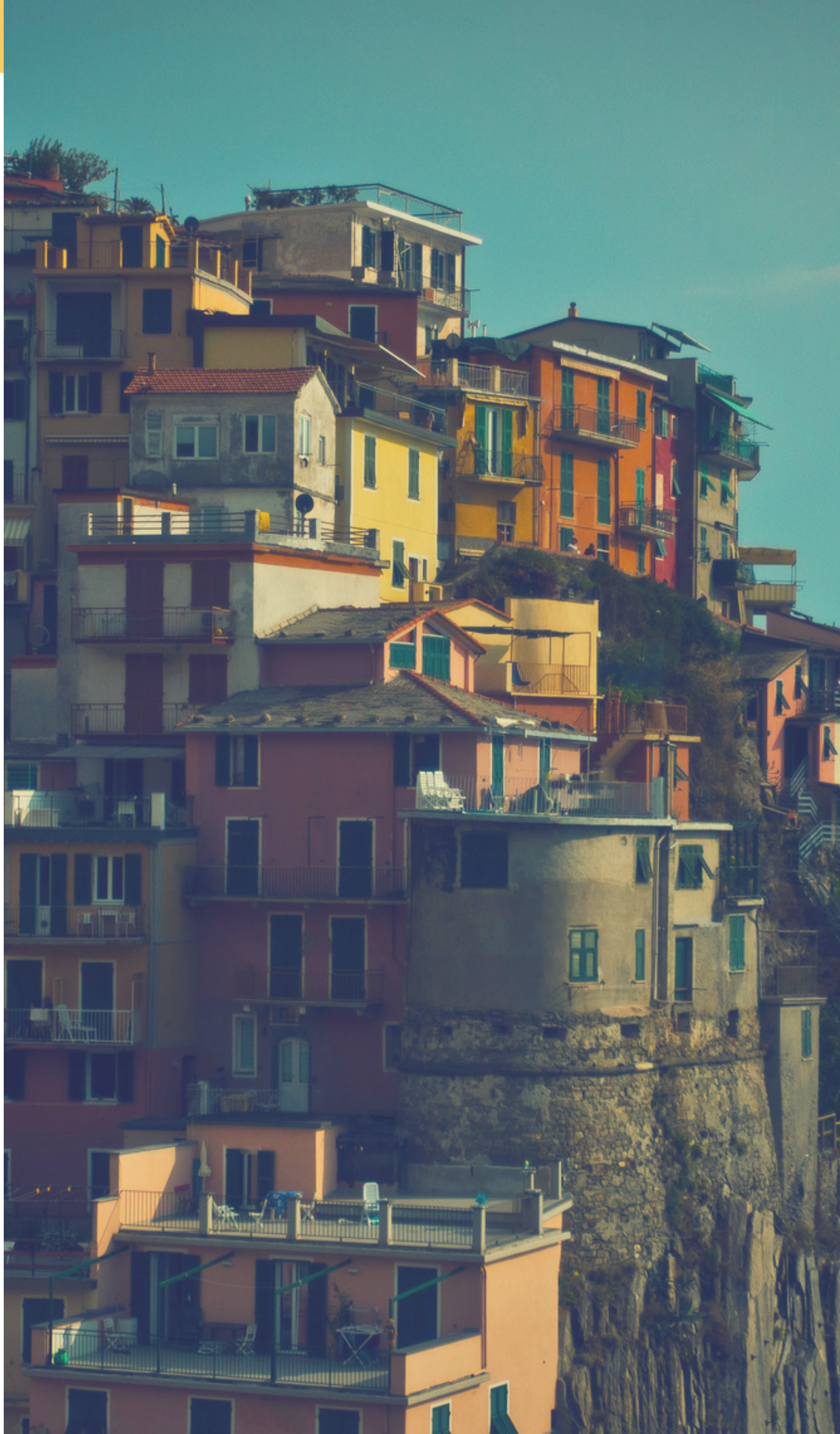
THE 246 BREAKTHROUGH

=

MAIN RESULT

THEOREM [CFG 2018]

"There are infinitely many prime sum graphs that have a Hamilton cycle"



IDEA

01

ASSUME there exists g with infinitely many prime pairs satisfying p′ - p =g.

Take g=12 for example.

There exist infinitely many (p, p′) with

$p = 12k' + 1$ or $p = 12k' + 5$ or

$p = 12k' + 7$ or $p = 12k' + 11$

02

MATCH prime pairs to gcd condition

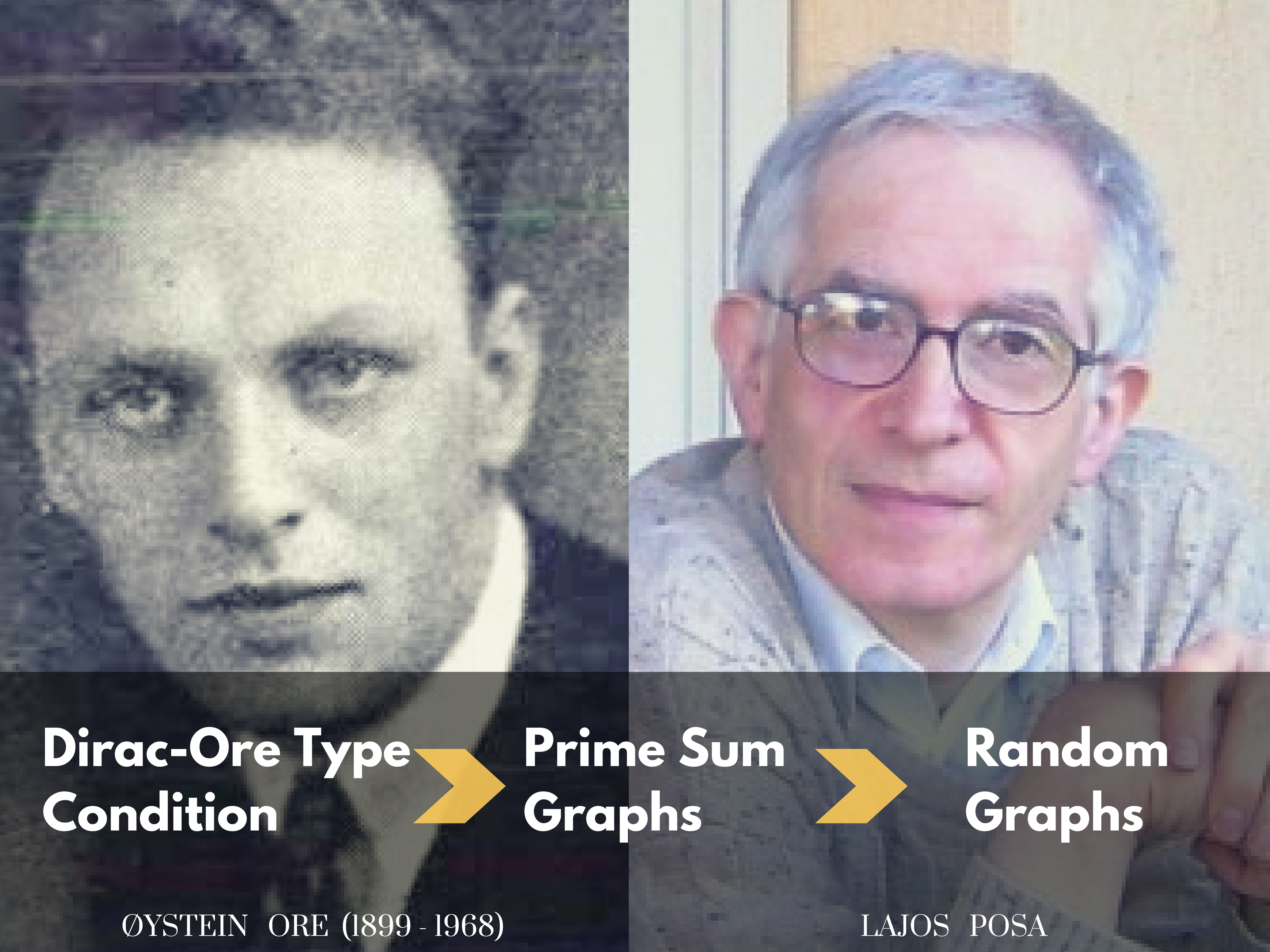
Form	(p_1, p_2)	gcd condition
$p = 12k' + 1$	(11,23)	$\gcd(6, 6k+1) = 1$
$p = 12k' + 5$	(7,19)	$\gcd(6, 6k+5) = 1$
$p = 12k' + 7$	(5,17)	$\gcd(6, 6k+1) = 1$
$p = 12k' + 11$	(1,13)	$\gcd(6, 6k+5) = 1$

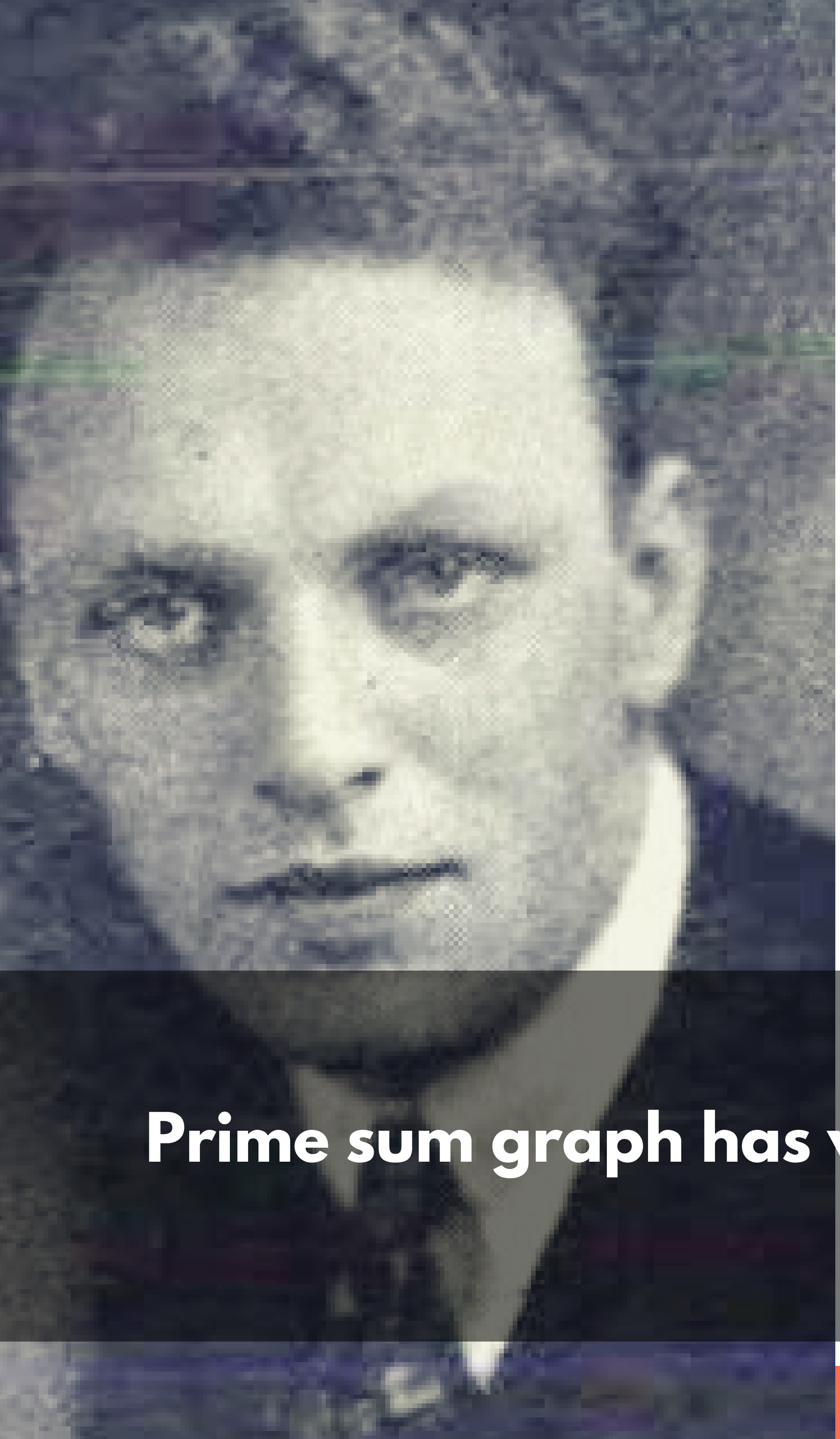
03

CHECK these steps for g=2, 4, ..., 246

There are total 6170 cases and this can be done by computers.


WHAT
RELEVANT

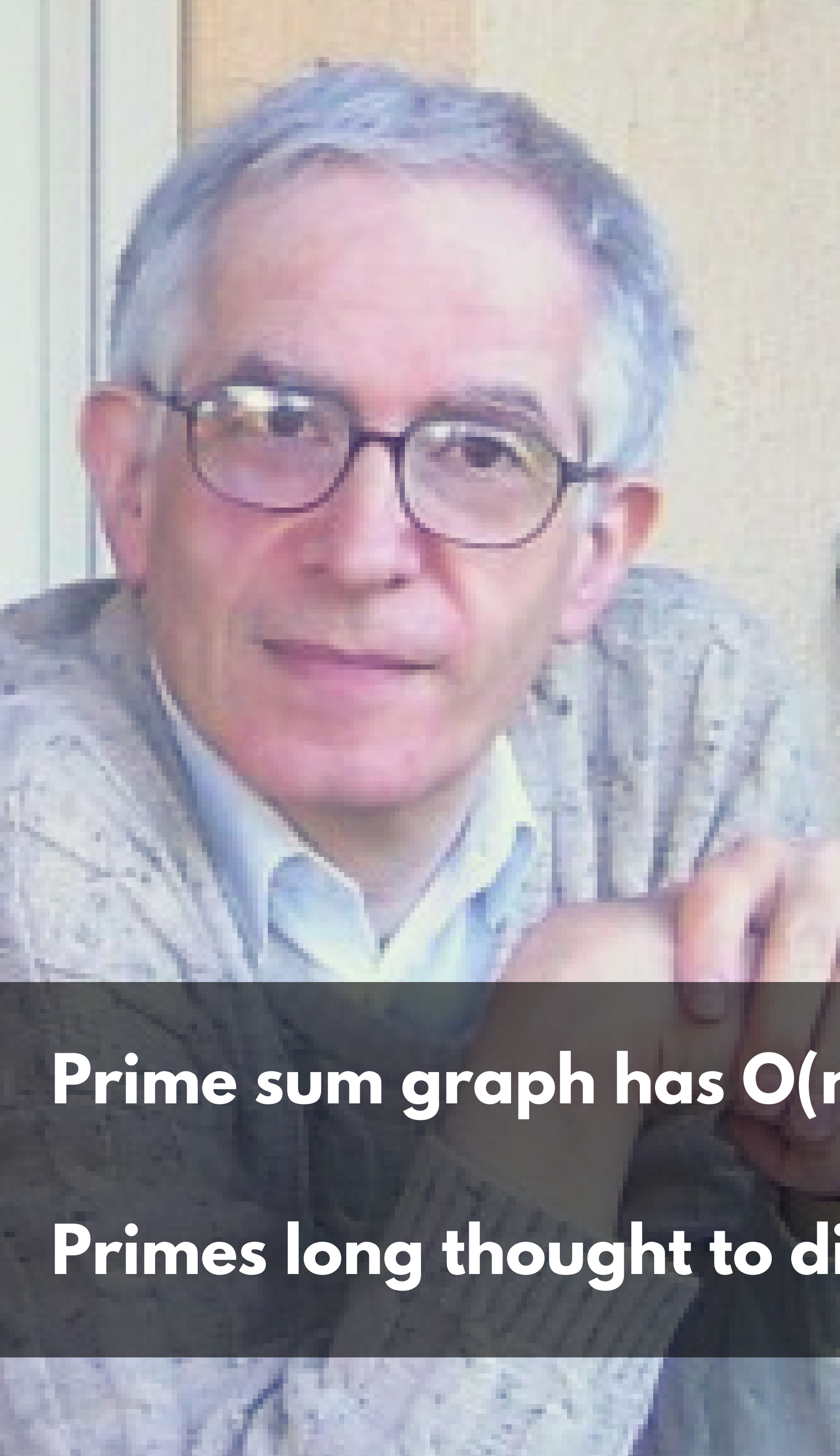

Dirac-Ore Type Condition
Prime Sum Graphs
Random Graphs
ØYSTEIN ORE (1899 - 1968)
LAJOS POSA



Dirac-Ore Type Condition

Graphs
with min. degree $n/2$
(with degree sum of 2 nonadj. vertices at least n)
have a Hamilton cycle

Prime sum graph has vertex degree $n/\log n << n/2$



Posa's THM. 1976

The probability
a random graph with n vertices
and cnlog n edges
for sufficiently large c
contains a Hamilton cycle
tends to 1 as n tends to infinity

Prime sum graph has O(n^2/log n) >> O(nlog n) edges

Primes long thought to distribute 'RANDOMLY' in a sense

We have verified the sufficient condition up to 100,000,000

99999976	13	31	99999989	100000007
99999978	11	61	99999989	100000039
99999980	59	101	100000039	100000081
99999982	7	67	99999989	100000049
99999984	5	139	99999989	100000123
99999986	3	53	99999989	100000039
[illegible]	[illegible]	61	100000007	100000049
[illegible]	17	223	100000007	100000213
[illegible]	[illegible]	[illegible]	[illegible]	100000081
[illegible]	[illegible]	[illegible]	[illegible]	100000037
99999996	11	241	100000007	100000237
99999998	41	233	100000039	100000231
100000000	7	73	100000007	100000073

CONTRIBUTION 01

There are infinitely many prime sum graphs that are Hamiltonian

CONJECTURE 02

The sufficient condition is always true?

If so

then we generalize Bertrand's postulate for even case

CHALLENGE 03

Generalized Bertrand's postulate?

" for each 2n, there is p with $2n < 2n+p < 4n$ such that p and 2n+p are prime "

Goldbach's Conjecture

$2n = p + p$

Best result by J.R. Chen

$2n = p + pp$

Generalized Bertrand's postulate $2n < 2n+p < 4n$

Implies $2n = p - p$, variant of Goldbach's conj.

Best result $2n = p - pp$ by J.R. Chen

Every prime sum graph of order 2n>2 is Hamiltonian

Who's Conjecture

Prime pyramids

Similarly, Margaret Kenney (1986) and Richard Guy (1993) studied the prime pyramid in which row n contains numbers 1, 2, ..., n, begins with 1, ends with n, and each adjacent pair summing to a prime.

Open Question:
Prove it for infinitely many n.

Erdős asked

Are there infinitely many primes p such that every even number <p-2 can be expressed as the difference between two primes each no more than p?

Example of p=13
10=13-3
8=11-3
6=11-5
4=7-3
2=5-3

THANK YOU

for your time!

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