On the generalized Alon-Frankl-Lovász theorem

Peng-An Chen

Department of Applied Mathematics
Taitung University

August 19, 2019

Outline

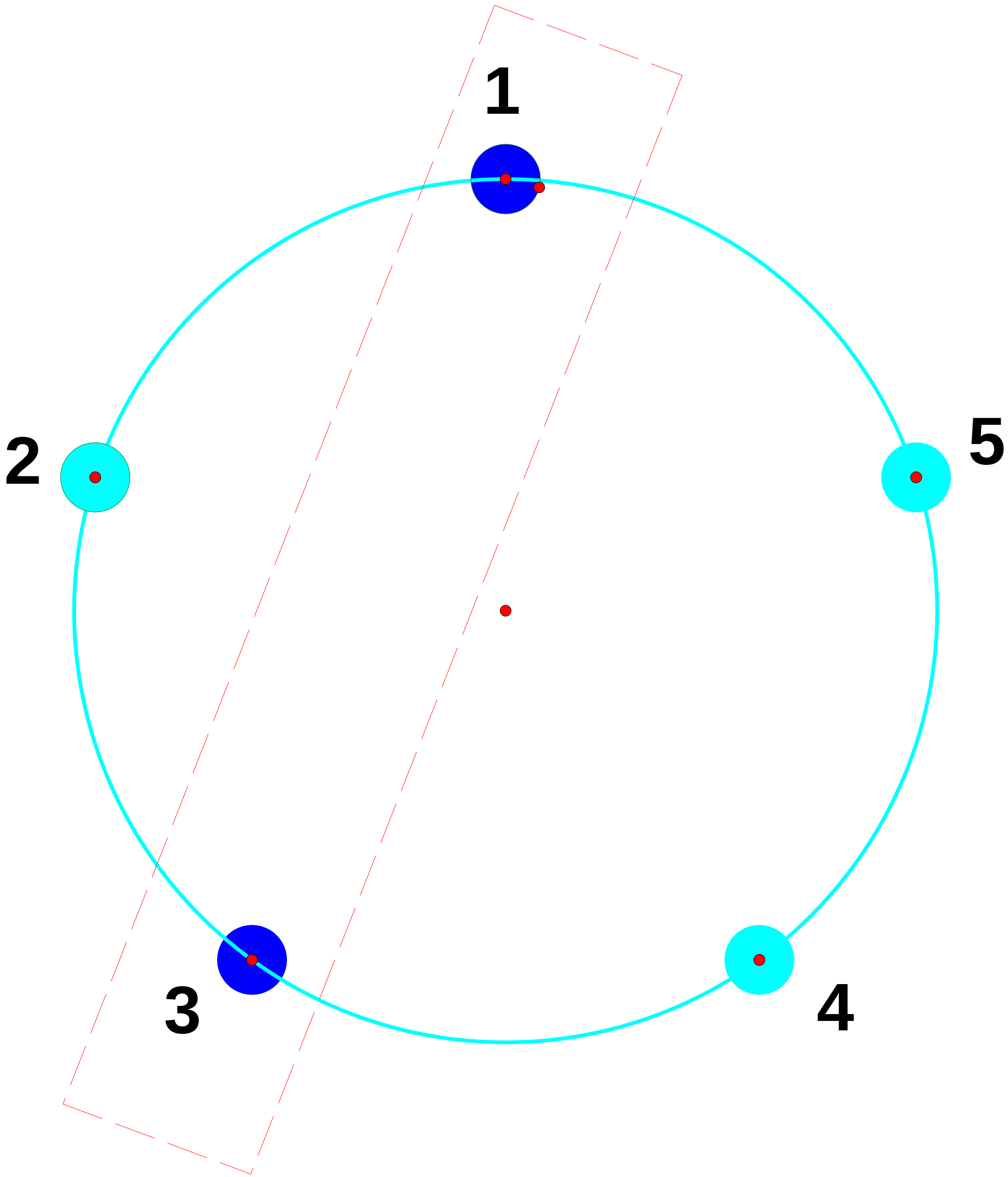
- We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$ of n integers.

- Denote $\binom{[n]}{k}$ as the collection of all k-subsets of $[n]$.

- We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$ of n integers.

- Denote $\binom{[n]}{k}$ as the collection of all k-subsets of $[n]$.

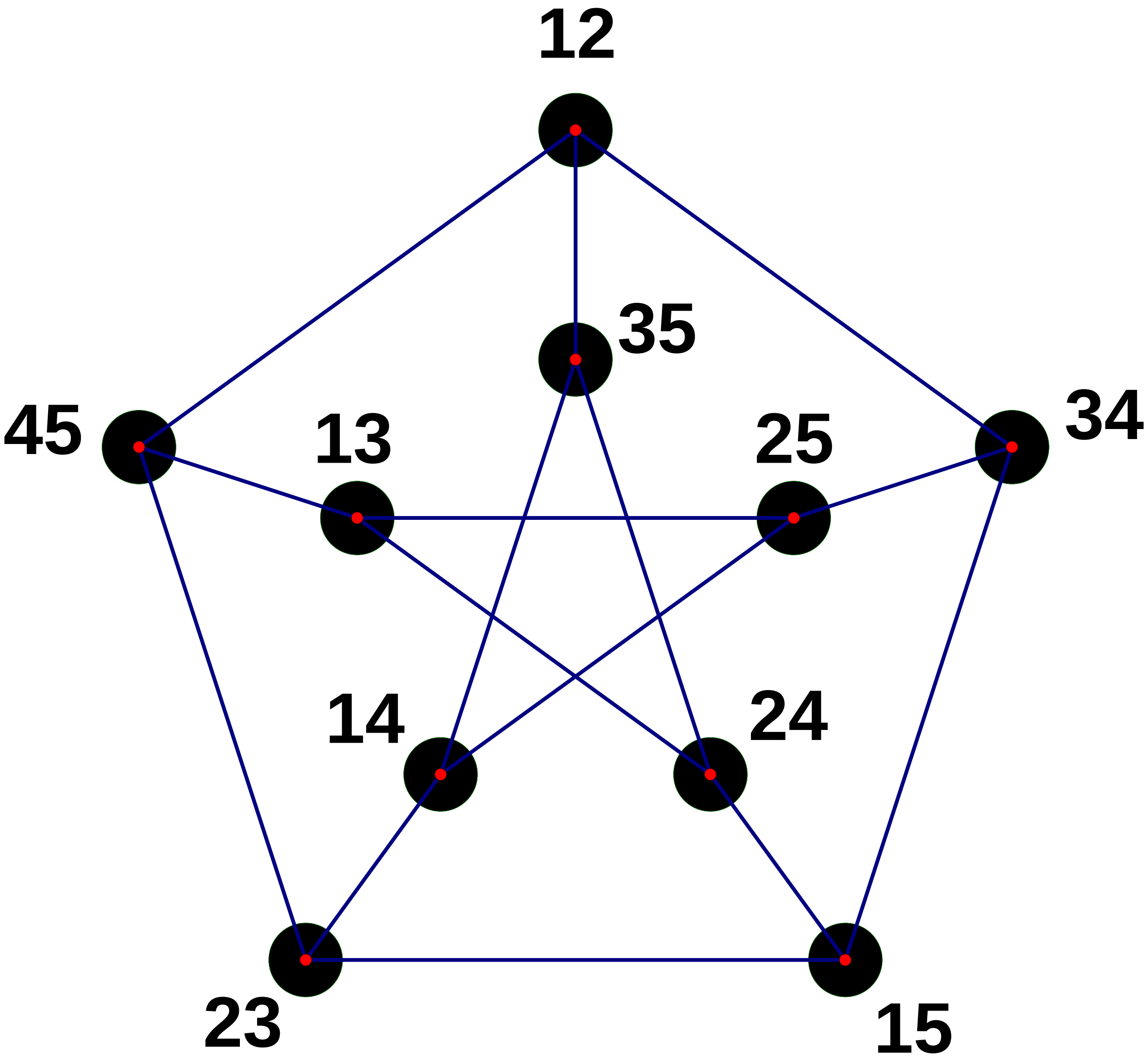
- For positive integers n, k and r, a k-subset $S \subseteq [n]$ is *r-stable* if $|S| = k$ and any two of its elements are at least "at distance r apart" on the n-cycle, that is, if $r \leq |i - j| \leq n - r$ for distinct $i, j \in S$. We denote by $\binom{[n]}{k}_{r\text{-}stab}$ the collection of all r-stable k-subsets.



{1,3} is a 2-stable subset of [5].

- The *r-uniform Kneser hypergraph* $\mathrm{KG}^r(n,k)$ is an r-uniform hypergraph which has $\binom{[n]}{k}$ as vertex set and whose edges are formed by the r-tuples of disjoint k-subsets of $[n]$.

- Choosing $r = 2$, we obtain the ordinary Kneser graph $\mathrm{KG}^2(n,k) = \mathrm{KG}\,(n,k)$.



$$KG^2(5,2) = KG(5,2)$$

- **The Kneser conjecture (1955)**:
  Let n and k be two positive integers with $n \geq 2k \geq 2$. Then

$$\chi(\mathrm{KG}\,(n,k)) = n - 2k + 2.$$

- **The Kneser conjecture (1955)**:
  Let n and k be two positive integers with $n \geq 2k \geq 2$. Then

$$\chi(\mathrm{KG}(n,k)) = n - 2k + 2.$$

- The Kneser conjecture (1955) was proved by **Lovász (1978)** using the **Borsuk-Ulam theorem**; all subsequent proofs, extensions and generalizations also relied on Algebraic Topology results.

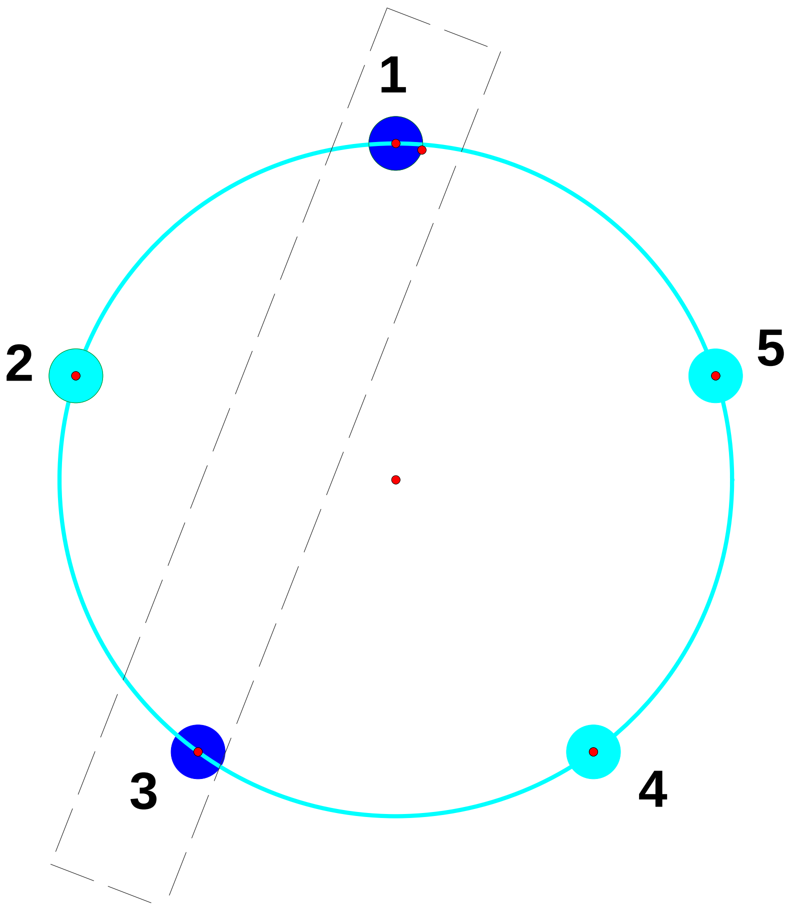
- **The Kneser conjecture (1955)**:
  Let n and k be two positive integers with $n \geq 2k \geq 2$. Then

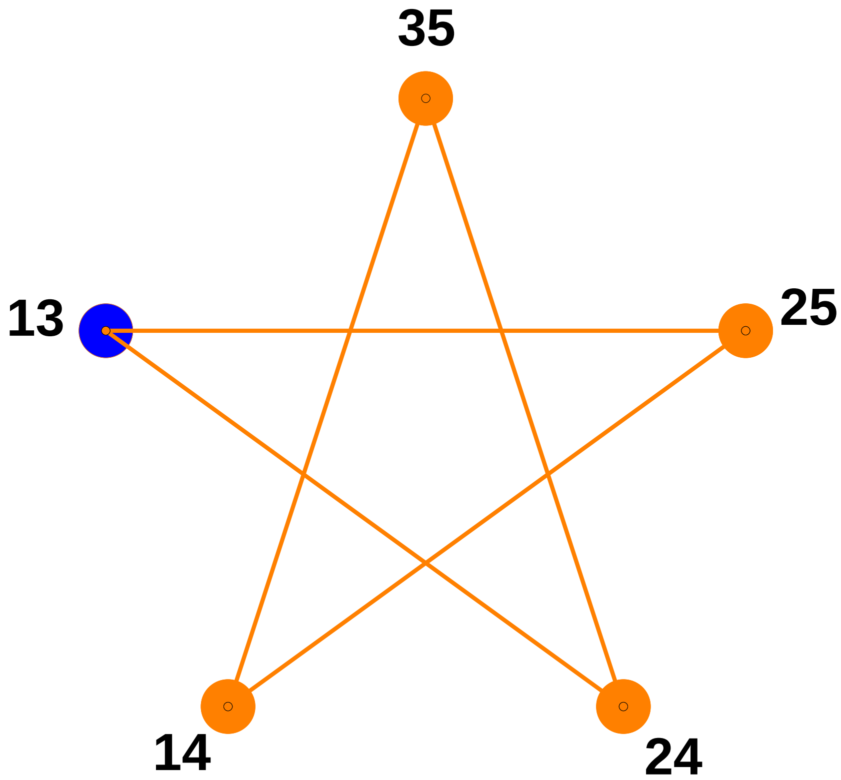
$$\chi(\mathrm{KG}\,(n,k)) = n - 2k + 2.$$

- The Kneser conjecture (1955) was proved by **Lovász (1978)** using the **Borsuk-Ulam theorem**; all subsequent proofs, extensions and generalizations also relied on Algebraic Topology results.

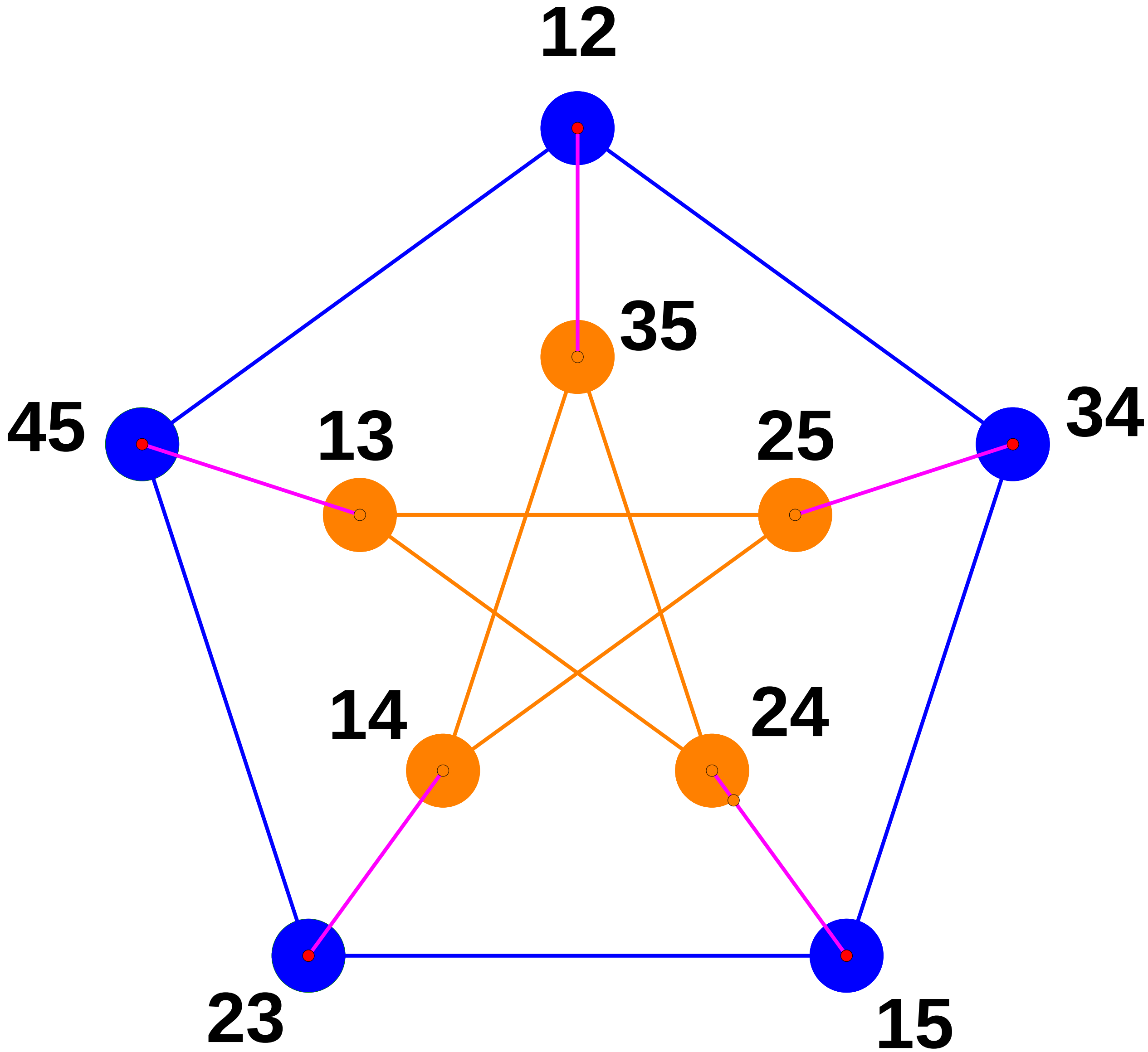
- **Matoušek (2004)** provided the first combinatorial proof of the Kneser conjecture via **Tucker's lemma** (1942).

- The *r-uniform r-stable Kneser hypergraph* $\mathrm{KG}^r(n,k)_{r\text{-}stab}$ is an r-uniform hypergraph which has $\binom{[n]}{k}_{r\text{-}stab}$ as vertex set and whose edges are formed by the r-tuples of disjoint r-stable k-subsets of $[n]$.





$KG^2(5,2)_{2-stab}$



$$KG^2(5,2) = KG(5,2)$$

- **Schrijver's theorem (1978)**:
  Let n and k be two positive integers with $n \geq 2k \geq 4$. Then

$$\chi(\mathrm{KG}^2(n,k)_{2\text{-}stab}) = n - 2k + 2 = \chi(KG(n,k)).$$

Outline

Erdős (1976), Alon, Frankl and Lovász (1986)

- **The Erdős conjecture (1976)**:
  For $n \geq rk$ and $r \geq 2$,

$$\chi(\mathrm{KG}^r(n,k)) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil .$$

Erdős (1976), Alon, Frankl and Lovász (1986)

- **The Erdős conjecture (1976)**:
  For $n \geq rk$ and $r \geq 2$,
  $$\chi\left(\mathrm{KG}^r(n,k)\right) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil.$$

- The conjecture settled by **Alon, Frankl and Lovász (1986)**.

Z_p-Tucker Lemma, Ziegler (2002)

Lemma

Let p be a prime, n, $m \geq 1$, and let

$$\begin{array}{rcl} \lambda : (Z_p \cap \{0\})^n \setminus \{0\}^n & \longrightarrow & Z_p \times [m] \\ X & \longmapsto & (\lambda_1(X), \lambda_2(X)) \end{array}$$

be a Z_p-equivariant map.
If $(p-1)m < n$, then there exist $X^{(1)} \subset X^{(2)} \subset \ldots \subset X^{(p)}$ such that $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = \lambda_2(X^{(p)})$, but with distinct signed $\lambda_1(X^{(i)}) \in Z_p$.

The Ziegler conjecture (2002)

- **The Ziegler conjecture (2002) for r-uniform r-stable Kneser hypergraph $\mathrm{KG}^r(n,k)_{r\text{-}stab}$:**
  For $n \geq rk$, $r \geq 2$, and $k \geq 2$,
  $$\chi\left(\mathrm{KG}^r\left(n,k\right)_{r\text{-}stab}\right) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil = \chi\left(\mathrm{KG}^r(n,k)\right).$$

Alon, Drewnowski and Łuczak (2009)

- **Alon, Drewnowski and Łuczak (2009)** settled the Ziegler conjecture in the particular case that $r = 2^q$ for arbitrary positive integer q.

Sani and Alishahi (2018)

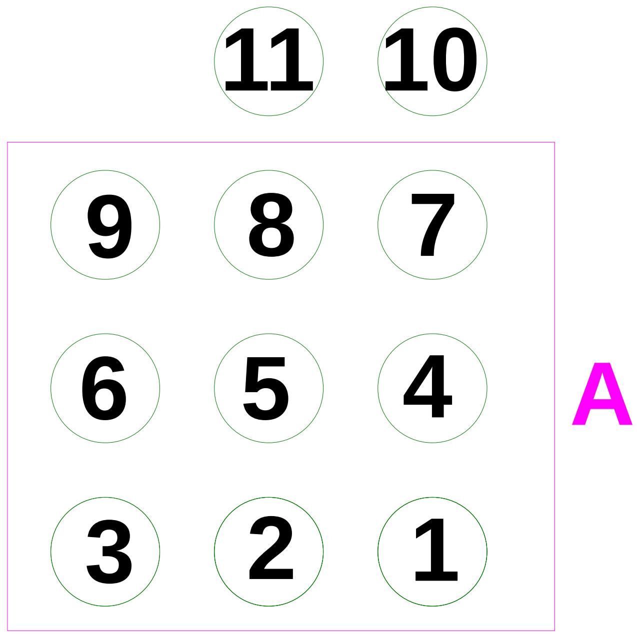
- Let n, k and r be positive integers with $k, r \geq 2$, and $n \geq rk$. For a set $A \subsetneq [n]$, define $\mathrm{KG}^r(n, k, A)$ to be a hypergraph whose vertices are k-element subsets $e \subset [n]$ with $e \not\subseteq A$, and whose hyperedges consists of r such sets that are pairwise disjoint.

- $\mathrm{KG}^r(n, k, A)$ is the subhypergraph of $\mathrm{KG}^r(n, k)$.

[11]

	11	10
9	8	7
6	5	4
3	2	1

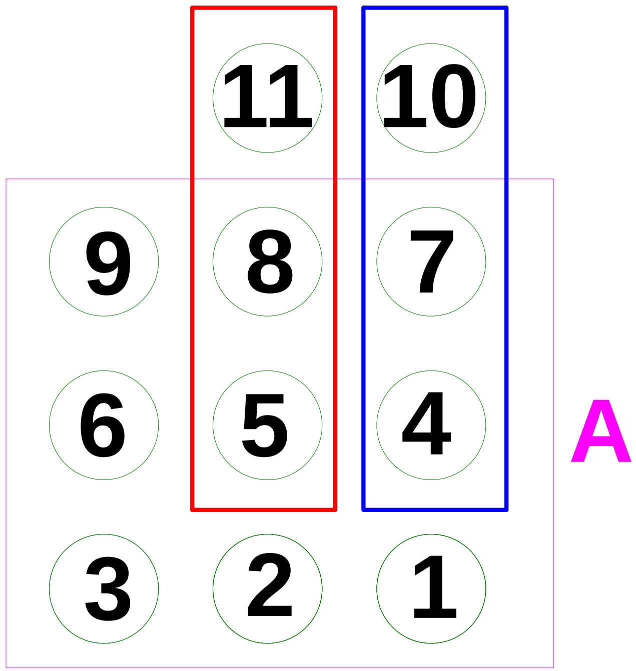
[11]



A is a proper subset of [11] and $|A| = 9$

$n = 11, r = 2, k = 3$ and $n \geq rk$

The hyperedge $e = \{\{5,8,11\}, \{4,7,10\}\}$



A is a proper subset of [11] and $|A| = 9$

Sani and Alishahi (2018)

- **Sani and Alishahi (2018) conjectured**:
  Let n, k and r be positive integers with $k, r \geq 2$, and $n \geq rk$. For a set $A \subsetneq [n]$,

$$\chi\left(\mathrm{KG}^r(n, k, A)\right) = \left\lceil \frac{n - \max\left(r(k-1), |A|\right)}{r-1} \right\rceil.$$

- Sani and Alishahi showed that the conjecture holds when $|A| \leq 2(k-1)$ or $|A| \geq rk - 1$, so the open case left is when $2k - 1 \leq |A| \leq rk - 2$.

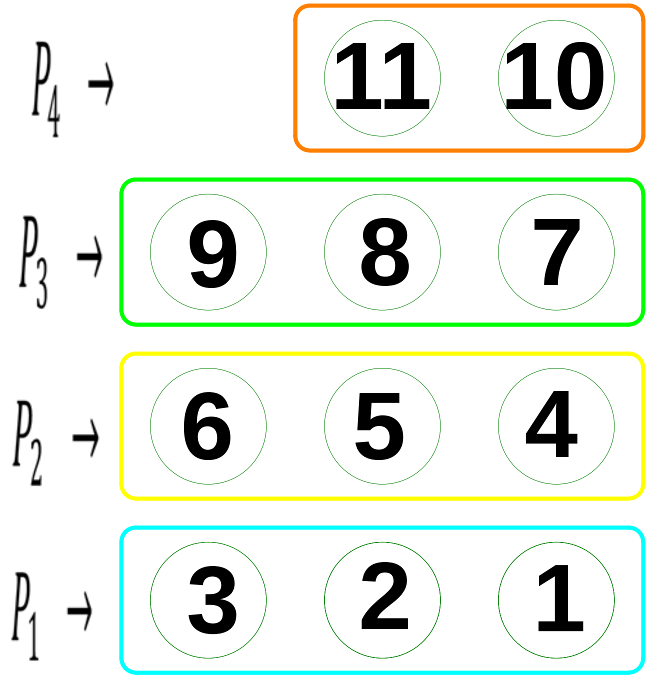
Frick et al. (2019)

- Let $\mathcal{P} = \{P_1, ..., P_\ell\}$ be a partition of $[n]$; denote by $\mathrm{KG}^r(n, k; \mathcal{P})$ (or by $\mathrm{KG}^r(n, k, P_1, ..., P_\ell)$) the hypergraph whose vertices are k-element subsets $e \subset [n]$ with $|e \cap P_i| \leq 1$ for all $i \in [\ell]$, and whose hyperedges consists of r such sets that are pairwise disjoint.

- $\mathrm{KG}^r(n, k; \mathcal{P})$ is a subhypergraph of $\mathrm{KG}^r(n, k, A)$ with $A \subseteq P_1 \cup \cdots \cup P_{k-1}$.

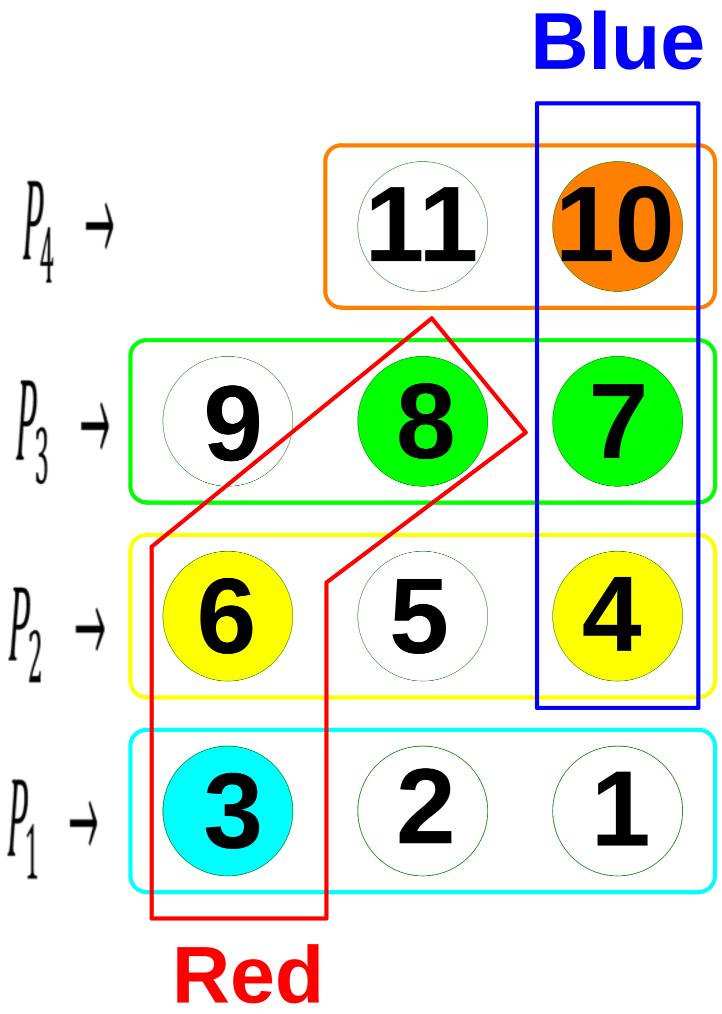
[11]

$\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ is the partition of $[11]$



$\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ is the partition of $[11]$

$n = 11, r = 2, k = 3$ and $n \geq rk$



The hyperedge $e = \{\{3,6,8\}, \{4,7,10\}\}$

Frick et al. (2019)

Theorem

Let $r \geq 2$, $k \geq 1$, and $n \geq rk$ be integers. Let $\mathcal{P} = \{P_1, ..., P_\ell\}$ be a partition of $[n]$ with $|P_i| \leq r-1$. Then

$$\chi\left(KG^r(n,k;\mathcal{P})\right) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil.$$

Frick et al. (2019)

Theorem

Let $r \geq 2$, $k \geq 1$, and $n \geq rk$ be integers. Let $\mathcal{P} = \{P_1, \dots, P_\ell\}$ be a partition of $[n]$ with $|P_i| \leq r-1$. Then

$$\chi\left(KG^r(n, k; \mathcal{P})\right) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil .$$

- Frick et al. (2019) proved some partial results of the Sani-Alishahi conjecture.

Frick et al. (2019)

- **Frick et al. (2019) conjectured**:
  Let $r \geq 2$, $k \geq 1$, and $n \geq rk$ be integers. Let $\mathcal{P} = \{P_1, \ldots, P_\ell\}$ be a partition of $[n]$ with $|P_i| \leq r$. Then

$$\chi\left(\mathrm{KG}^r(n, k; \mathcal{P})\right) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil.$$

Chen (2019)

- We obtain the following result via Z_p-**Tucker lemma**.

Theorem

Let n, k and r be positive integers with $k, r \geq 2$, and $n \geq rk$. For a set $A \subsetneq [n]$,

$$\chi\left(KG^r(n,k,A)\right) = \left\lceil \frac{n - \max\left(r(k-1), |A|\right)}{r-1} \right\rceil.$$

Chen (2019)

- We obtain the following result via Z_p-**Tucker lemma**.

Theorem

Let $r \geq 2$, $k \geq 1$, and $n \geq rk$ be integers. Let $\mathcal{P} = \{P_1, \ldots, P_\ell\}$ be a partition of $[n]$ with $|P_i| \leq r$. Then

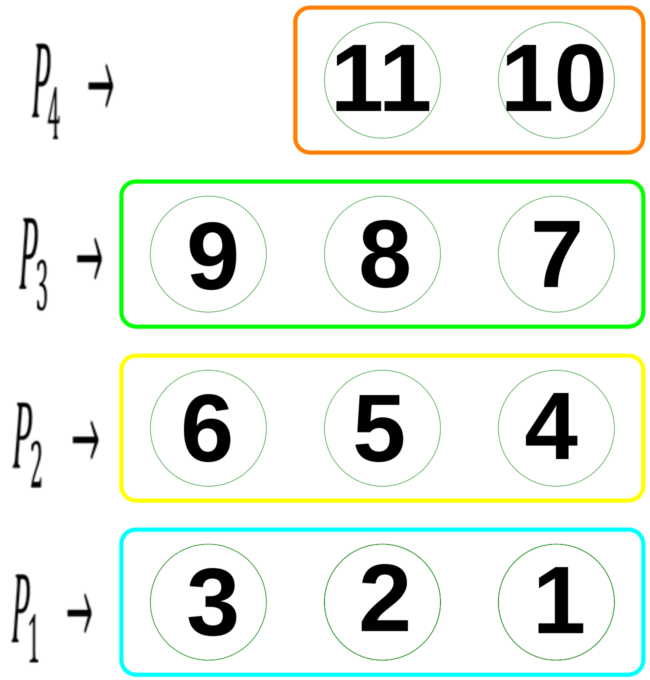
$$\chi\left(KG^r(n,k;\mathcal{P})\right) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil.$$

The Ziegler conjecture (2002)

- $\mathrm{KG}^r(n,k)_{r\text{-}stab}$ is a subhypergraph of $\mathrm{KG}^r(n,k;\mathcal{P})$ after possibly reordering $[n]$ to make each part P_i consist of consecutive elements.
- **The Ziegler conjecture is still open.**
  For $n \geq rk$, $r \geq 2$, and $k \geq 2$,

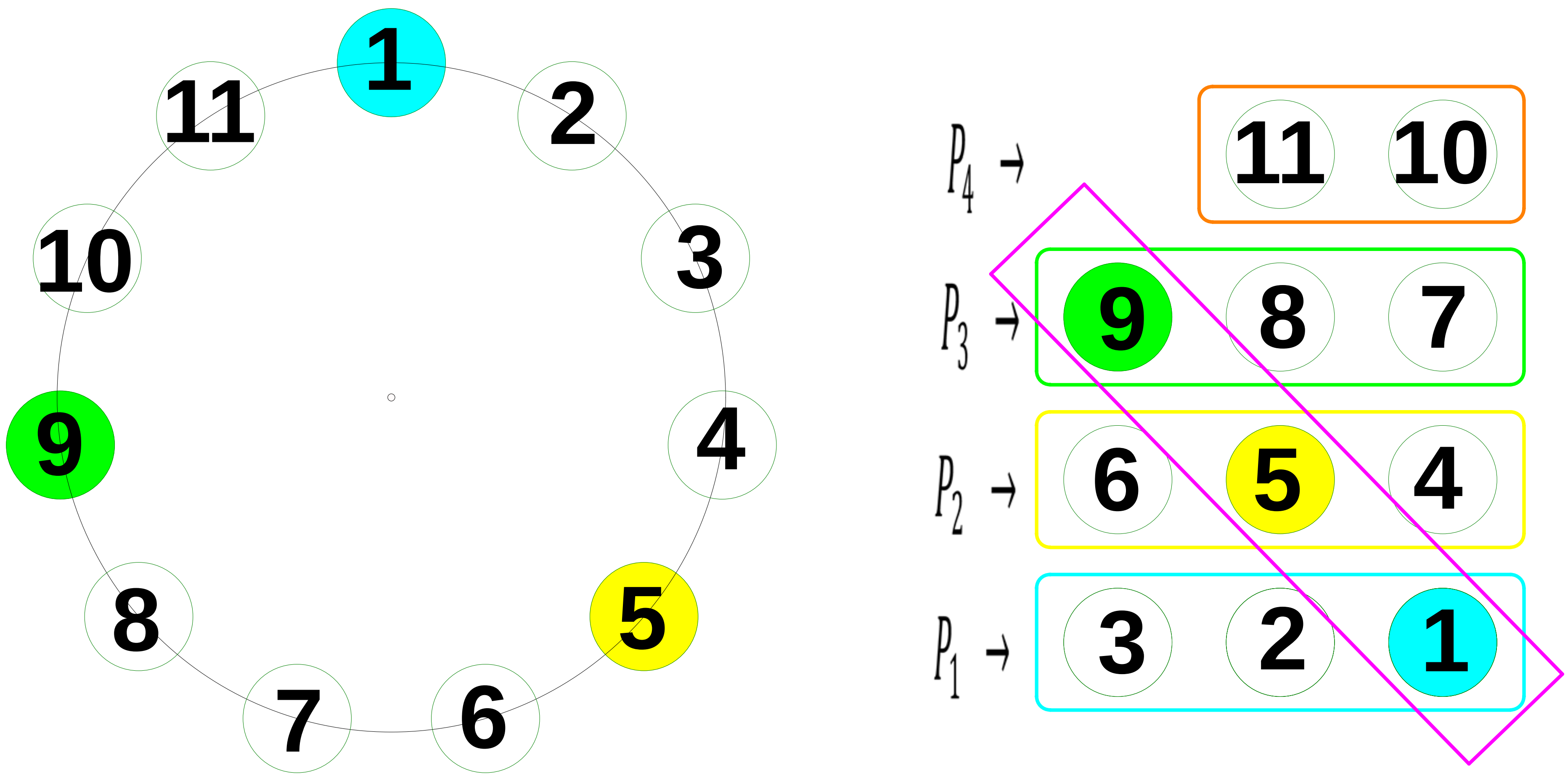
$$\chi\left(\mathrm{KG}^r\left(n,k\right)_{r\text{-}stab}\right) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil = \chi\left(\mathrm{KG}^r(n,k)\right).$$

$\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ is the partition of $[11]$



$\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ is the partition of $[11]$

The vertex $\{1,5,9\}$ is a 3-stable 3-subset of $[11]$



Hamid Reza Daneshpajouh (2019)

- Let n, k, r, s be non-negative integers where $n \geq r(k-1)+1$, $k > s \geq 0$ and $r \geq 2$. Define $\mathrm{KG}^r(n,k,s)$ to be a hypergraph whose vertices are k-element subsets $e \subset [n]$ and edge set $E\left(\mathrm{KG}^r(n,k,s)\right) = \left\{\{e_1,\ldots,e_r\} : e_i \in \binom{[n]}{k} \text{ and } |e_i \cap e_j| \leq s \text{ for all } i \neq j\right\}$

Hamid Reza Daneshpajouh (2019)

- Let n, k, r, s be non-negative integers where $n \geq r(k-1)+1$, $k > s \geq 0$ and $r \geq 2$. Define $\mathrm{KG}^r(n,k,s)$ to be a hypergraph whose vertices are k-element subsets $e \subset [n]$ and edge set $E\left(\mathrm{KG}^r(n,k,s)\right) = \left\{\{e_1,\ldots,e_r\} : e_i \in \binom{[n]}{k} \text{ and } |e_i \cap e_j| \leq s \text{ for all } i \neq j\right\}$

- $\mathrm{KG}^r(n,k,0)$ is the Kneser hypergraph $\mathrm{KG}^r(n,k)$.
- $\mathrm{KG}^2(n,k,k-1)$ is the complete graph $\mathrm{K}_{\binom{n}{k}}$.

Hamid Reza Daneshpajouh (2019)

- Let n, k, r, s be non-negative integers where $n \geq r(k-1)+1$, $k > s \geq 0$ and $r \geq 2$. Define $\mathrm{KG}^r(n,k,s)$ to be a hypergraph whose vertices are k-element subsets $e \subset [n]$ and edge set $E\left(\mathrm{KG}^r(n,k,s)\right) = \left\{\{e_1,\ldots,e_r\} : e_i \in \binom{[n]}{k} \text{ and } |e_i \cap e_j| \leq s \text{ for all } i \neq j\right\}$

- $\mathrm{KG}^r(n,k,0)$ is the Kneser hypergraph $\mathrm{KG}^r(n,k)$.
- $\mathrm{KG}^2(n,k,k-1)$ is the complete graph $\mathrm{K}_{\binom{n}{k}}$.

- The chromatic number of $\mathrm{KG}^r(n,k,s)$ were studied by Frankl and Füredi (1985, 1986) for fixed k and s.

Hamid Reza Daneshpajouh (2019)

- Daneshpajouh obtains the following result via Z_p-**Tucker lemma**.

Theorem

Let n, k, r, s be non-negative integers where $n \geq r(k-1)+1$, $k > s \geq 0$ and $r \geq 2$,

$$\chi\left(KG^r(n,k,s)\right) \geq \left\lceil \frac{n - r(k-s-1)}{r-1} \right\rceil .$$

Hamid Reza Daneshpajouh (2019)

- Daneshpajouh obtains the following result via Z_p-**Tucker lemma**.

Theorem

Let n, k, r, s be non-negative integers where $n \geq r(k-1)+1$, $k > s \geq 0$ and $r \geq 2$,

$$\chi\left(KG^r(n,k,s)\right) \geq \left\lceil \frac{n - r(k-s-1)}{r-1} \right\rceil.$$

- $\chi\left(\mathrm{KG}^2(n,k,k-1)\right) = \binom{n}{k}$.

Hamid Reza Daneshpajouh (2019)

- Let n, k, r be non-negative integers where $n \geq r(k-1)+1$ and $r \geq 2$,

$$\chi(\mathrm{KG}^r(n,k)) = \chi(\mathrm{KG}^r(n,k,0)) \geq \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil.$$

Hamid Reza Daneshpajouh (2019)

- Let n, k, r be non-negative integers where $n \geq r(k-1)+1$ and $r \geq 2$,

$$\chi(\mathrm{KG}^r(n,k)) = \chi(\mathrm{KG}^r(n,k,0)) \geq \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil .$$

- It is of interest to know when the inequality obtained by Daneshpajouh is sharp.

Thank you for your attention!!!