# Disjoint cycles in digraphs

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1. Notation
2. Introduction and Main Theorem
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## 1. Notation

- A graph $G = (V, E)$: $V$ vertex set, $E$ edge set.

  Let $n$ be the order of $G$ for simplicity, i.e. $n = |V|$.

  $\delta(G)$: the minimum degree of $G$.

- A digraph $D = (V, A)$: $V$ vertex set, $A$ arc set.

  $\delta^+(D)$: the minimum out-degree,

  $\delta^-(D)$: the minimum in-degree of $D$.

- The semi-degree of $D$ is $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$.

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- In a digraph: a cycle is always **directed**.

- A **tournament** is a digraph $T$ such that for any two distinct vertices $x$ and $y$, exactly one of the ordered pairs $(x, y)$ and $(y, x)$ is an arc of $T$.
- A set of subgraphs of $G$ or $D$ is said to be **vertex-disjoint**( briefly **disjoint** ) if no two of them have any common vertex in $G$ or $D$.
- **A 2-factor** of a graph $G$ is a spanning subgraph of $G$ such that each component is a cycle.
  **A hamiltonian cycle** is a 2-factor with exactly one component.
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## 2. Introduction and Main Theorem

For a graph $G$, if $e(G) \geq n$, or $\delta(G) \geq 2$, then $G$ contains a cycle.

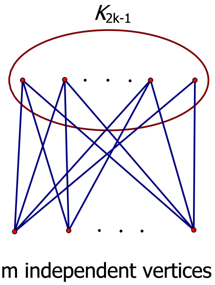
**How about two disjoint cycles?**

- P. Erdős, L. P$\acute{o}$sa, 1962
  For every graph $G$, if $n \geq 6$ and $e(G) \geq 3n - 6$, then $G$ has 2 disjoint cycles or isomorphic to $K_3 + (n-3)K_1$.

## Given an integer $k \geq 1$, how about $k$ disjoint cycles?

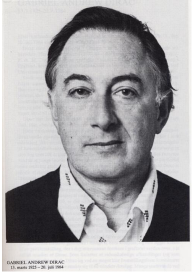
- K. Corrádi and A. Hajnal, 1963
  For any positive integer $k$ and any graph $G$, if $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ has $k$ disjoint cycles.

Sharpness of degree condition is given by $G = K_{2k-1} + mK_1$.




$$G = K_{2k-1} + mK_1$$

**What is the degree condition for the disjoint cycles with the same length?**




- G. A. Dirac, 1963
  For any positive integer $k$ and any graph $G$, if $n \geq 3k$ and $\delta(G) \geq (n+k)/2$, then $G$ contains k disjoint triangles.

- The papers in this topic can be found in **[Degree Conditions for the Existence of Vertex-Disjoint Cycles and Paths: A Survey, Graphs and Combinatorics (2018) 34:1 – 83. ]**

For a digraph $D$, if $\delta^+(D) \geq 1$, then $D$ contains a cycle.

Given a positive integer $k$ and a digraph $D$,
what is the degree condition for $k$ disjoint cycles in a digraph?

**Conjecture 2.1 [Bermond, Thomassen, J. Graph Theory 5 (1) (1981) 1-43.]**

For $k \geq 1$ and any digraph $D$, if $\delta^+(D) \geq 2k-1$, then $D$ contains $k$ disjoint cycles.

- Thomassen proved the case $k = 2$ in 1983 (Combinatorica);
- Lichiardopol et al. proved the case $k = 3$ in 2009 (SIAM Discrete Math.);
- **Conjecture 2.1 remains open for $k \geq 4$.**

Thomassen further proposed the conjecture on disjoint cycles with the same length.

**Conjecture 2.2 [Thomassen, Combinatorica 3 (3-4) (1983) 393-396.]**

For each natural $k$ and any digraph $D$, there exists $g(k)$, if $\delta^+(D) \geq g(k)$, then $D$ contains $k$ pairwise disjoint cycles of the same length.

In 1996, Alon disproved the conjecture.

**Theorem 2.3 [Alon, J. Combin. Theory Ser. B 68 (2) (1996) 167-178.]**

For every integer $r$, there exists a digraph $D$ such that $\delta^+(D) = r$, but $D$ contains no two edge-disjoint cycles of the same length (and hence, of course, no two vertex-disjoint cycles of the same length).

Although Conjecture 2.2 is not true for general digraphs, Lichiardopol believed that it is correct for tournaments. He raised the following conjecture.

**Conjecture 2.4 [Lichiardopol, Discrete Math. 310 (19) (2010) 2567-2570.]**

For every tournament $T$, if $\delta^+(T) \geq (q-1)k-1$, then $T$ contains $k$ disjoint cycles of length $q$, where $q \geq 3$ and $k \geq 1$.

- In the same paper, Lichiardopol himself proved that if both the minimum out-degree and in-degree are at least $(q-1)k-1$, i.e. $\delta^0(T) \geq (q-1)k-1$, then $T$ contains $k$ disjoint cycles of length $q$.

- In 2013, Jensen, Bessy and Thomass$\acute{e}$ proved Conjecture 2.4 for the case $q = 3$ (A special case of Bermond-Thomassen conjecture).
- We confirm Lichiardopol's Conjecture for $q \geq 4$.

**Main Theorem 1 [F. Ma, Y. Wang and Y, 2019+]**

Conjecture 2.4 is correct.
That is, for any integer $q \geq 4$, $k \geq 1$ and every tournament $T$, if $\delta^+(T) \geq (q-1)k-1$, then it contains $k$ disjoint cycles of length $q$.

We also improved Lichiardopol's theorem on semi-degree condition significantly by proving the following theorem.

**Main Theorem 2 [F. Ma and Y, Applied Mathematics and Computation 347 (2019) 162-168.]**

For integers $k \geq 1$ and $q \geq 4$, every tournament $T$ with $\delta^0(T) \geq (q-1)k - 1$ contains $f(q)k - 2q$ disjoint cycles of length $q$, where $f(q) = \frac{6q^2-16q+10}{3q^2-3q-4}$.

### Note

- $f(q) > 1$, when $q \geq 4$
- $f(q)$ tends to 2, when $q$ tends to infinity.

## 3. Sketch of Main Theorem 1

Property 1. Every tournament has a hamiltonian path.

Property 2. Every strong tournament is vertex pancyclic.

- We prove Main Theorem by induction on $k$.
- When $k = 1$, then $\delta^+(T) \geq q - 2$.
  Let $P = (v_n \cdots v_1)$ be a hamiltonian path of $T$.

  Consider $v_1$, since $d^+(v_1) \geq q - 2$ and all its out-neighbours are on $P$, there exists $v_i$ with $i \geq q$ satisfying $v_1 v_i \in A$.
- So there is a cycle $v_i v_{i-1} \cdots v_1 v_i$ of length at least $q$. By property of pancyclicity, $T$ contains a $q$-cycle (a cycle of length $q$).

- Suppose that Main Theorem is correct for $k-1$,
  i.e. if $\delta^+(T) \geq (q-1)(k-1)-1$, then $T$ contains $k-1$ disjoint $q$-cycles.
- Now we consider the case $k$.
  Since in this case $\delta^+(T) \geq (q-1)k-1 > (q-1)(k-1)-1$, $T$ contains $k-1$ disjoint $q$-cycles.
- Using the following three auxiliary theorems, we may show that $T$ contains $k$ disjoint $q$-cycles,

**Theorem 3.1**

Let $q \geq 9$ and $k \leq q+1$. For every collection $\mathcal{F}$ of $k-1$ disjoint $q$-cycles of $T$, there exists a collection of $k$ disjoint $q$-cycles in $T$ which intersects $T \setminus \mathcal{F}$ on at most $3q$ vertices.

**Theorem 3.2**

Let $q \geq 10$ and $k \geq 3.3098\sqrt{q}$. For every collection $\mathcal{F}$ of $k-1$ disjoint $q$-cycles of $T$, there exists a collection of $k$ disjoint $q$-cycles in $T$ which intersects $T \setminus \mathcal{F}$ on at most $3q$ vertices.

• Since $3.3098\sqrt{q} \leq q+2$ when $q \geq 8$, Theorems 3.1 and 3.2 imply Main Theorem for $q \geq 10$.

**Theorem 3.3**

When $4 \leq q \leq 9$, we can refine Theorem 3.2 with "$|\mathcal{F} \cap (T \setminus \mathcal{F})| \leq 3q-6$".
That is, For every collection $\mathcal{F}$ of $k-1$ disjoint $q$-cycles of $T$, there exists a collection of $k$ disjoint $q$-cycles in $T$ which intersects $T \setminus \mathcal{F}$ on at most $3q-6$ vertices.

# 3. Sketch of Theorem 3.1

Let $\mathcal{F} = \{C_1, \ldots, C_{k-1}\}$ be a collection of $k-1$ disjoint $q$-cycles, $P = (u_r \cdots u_2 u_1)$ be a hamiltonian path of $T \setminus \mathcal{F}$.

We design the following pseudo algorithm.

**step 0.** Set $\mathcal{F} := \{C_1, \ldots, C_{k-1}\}$, $P := (u_r \cdots u_2 u_1)$ and $h := 0$. Let $C = (u_{j-1} u_{j-2} \cdots u_1 u_{j-1})$ be the longest cycle of $T[V(P)]$ through vertex $u_1$, if it exists.

**step 1.** If $C$ does not exist, then we construct a collection $\mathcal{F}'$ of $k-1$ disjoint $q$-cycles and a new hamiltonian path $P' = (v_r \cdots v_2 v_1)$ in $T \setminus \mathcal{F}'$ such that $|V(\mathcal{F}') \cap V(P)| \leq 2$ and $T \setminus \mathcal{F}'$ contains a cycle $C'$ through vertex $v_1$. Set $\mathcal{F} := \mathcal{F}'$, $P := P'$, $C := C'$ and $h := h+1$. Otherwise, go to step 2.

# 3. Sketch of Theorem 3.1

**step 2.** Construct a collection $\mathcal{F}'$ of $k-1$ disjoint $q$-cycles and a new hamiltonian path $P' = (v_r \cdots v_2 v_1)$ in $T \setminus \mathcal{F}'$ such that $|V(\mathcal{F}') \cap V((P))| \leq 2$ and $T \setminus \mathcal{F}'$ contains a cycle $C'$ with $|C'| \geq |C|+1$ through vertex $v_1$. Set $\mathcal{F} := \mathcal{F}'$, $P := P'$, $C := C'$ and $h := h+1$.

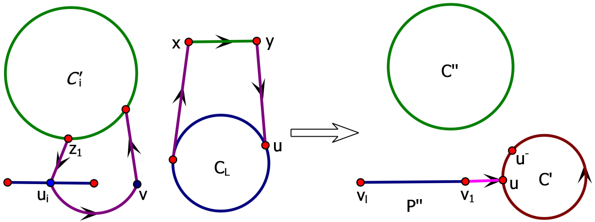



Fig.1

**step 3.** If $|C| \leq q-1$, then do step 2 recursively until $|C| \geq q$. Otherwise, output $\mathcal{F} \cup \{C\}$ and $h$.

# 3. Sketch of Theorem 3.1

- After step 3, $|C| \geq q$, by pancyclicity, there is a $q$-cycle $C'$. This cycle and $\mathcal{F}$ form a collection of $k$ disjoint $q$-cycles.
- Now we prove that the vertex number of $|(V(\mathcal{F} \cup \{C'\})) \cap V(P)|$ is at most $3q$ by using the algorithm.
- At each iteration of step 1 and step 2, we add at most two vertices outside $\{C_1, \ldots, C_{k-1}\}$ into $\mathcal{F}$. It follows by $h \leq q-2$ that

$$|V(\mathcal{F}) \cap V((T \setminus \{C_1, \ldots, C_{k-1}\}))| \leq 2h \leq 2q-4.$$

Therefore,

$$|(\mathcal{F} \cup \{C'\}) \cap (T \setminus \{C_1, \ldots, C_{k-1}\})| \leq 2q-4+q = 3q-4.$$

Thus Theorem 3.1 holds.

## Sketch of Theorem 3.2

Let $\mathcal{F} = \{C_1, \ldots, C_{k-1}\}$ be a collection of $k-1$ disjoint $q$-cycles and let $P = (u_r \cdots u_2 u_1)$ be a hamiltonian path of $T \setminus \mathcal{F}$. Partition $P$ by letting $U_1 = \{u_1, \ldots, u_{q+1}\}$, $S = \{u_{q+2}, \ldots, u_{4q-5}\}$ and $U_2 = V(P) \setminus (U_1 \cup S)$. That is, $U_1$ is the set of the last $q+1$ vertices on $P$ and $S$ is the last $3q-6$ vertices on $P \setminus U_1$.

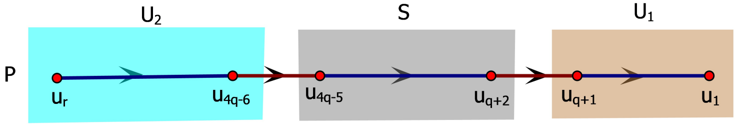



Fig.2 Partition of P

# 3. Sketch of Theorem 3.2

Denote by $\mathcal{I}$ the set of $q$-cycles that receive at least $q^2$ arcs each from $U_1$, by $\mathcal{O}$ the set of $q$-cycles that send at least $6q-1$ arcs each to $U_2$ and $\mathcal{R} = \mathcal{F} \setminus (\mathcal{I} \cup \mathcal{O})$. Furthermore, $i, o$ and $r$, respectively, denote the size of $\mathcal{I}, \mathcal{O}$ and $\mathcal{R}$.

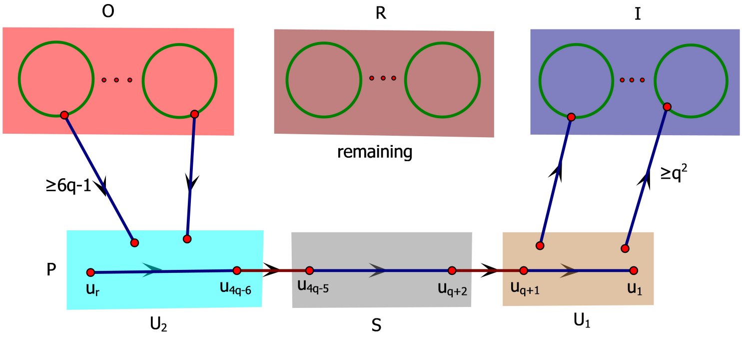



Fig.3 Partition of F

# 3. Sketch of Theorem 3.2

Now we estimate the lower and upper bound of the number of arcs leaving from $\mathcal{F} \setminus \mathcal{O} = \mathcal{I} \cup \mathcal{R}$.

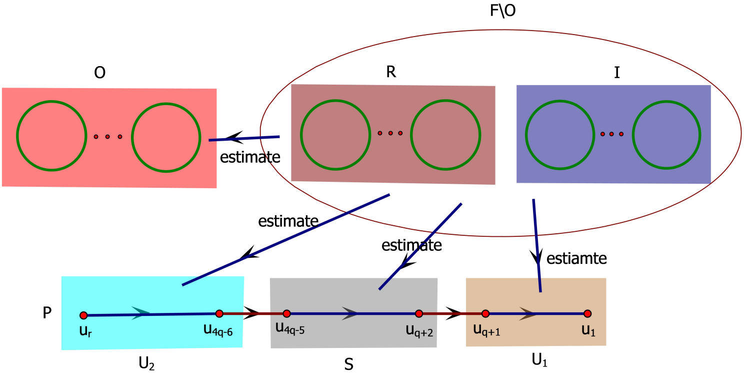



Fig.3 Partition of F

## 3. Sketch of Theorem 3.2

**First**, since $\delta^+(T) \geq (q-1)k-1$,

$$d^+(\mathcal{I} \cup \mathcal{R}) \geq q(i+r)((q-1)k-1) - \frac{1}{2}q(i+r)(q(i+r)-1).$$

**On the other hand**, we bound the number of arcs from $\mathcal{I}$ to $\mathcal{O}$ and $\mathcal{R}$ to $\mathcal{O}$, from $\mathcal{I}$ to $U_2$ and $\mathcal{R}$ to $U_2$, from $\mathcal{I} \cup \mathcal{R}$ to $S$ and $U_1$.

$$d^+(\mathcal{I} \cup \mathcal{R}) \leq (q^2-q+2)io + q^2ro + (3q-1)i + (6q-2)r + (3q-6)q(i+r)$$

$$+q(i+r)(q+1) - \alpha + (q^2-q-2)o,$$

where $\alpha = (q+1)((q-1)k-1) - \frac{1}{2}(q+1)q - \frac{1}{2}(q-2)(q-3)$.

## 3. Sketch of Theorem 3.2

So we get

$$ao^2 + bo + c < 0, \tag{1}$$

where

$$a = \frac{1}{2}q^2 + q - 2, \ b = ((q - q^2)k) + (3q^2 + \frac{13}{2}q - 9)$$

and

$$c = (\frac{1}{2}q^2 - q)k^2 + (1 - \frac{1}{2}q - 3q^2)k + (\frac{5}{2}q^2 + \frac{11}{2}q - 25).$$

Obviously, $a = \frac{1}{2}q^2 + q - 2 > 0$. Inequality (1) admits solution for $o$ only if

$$\Delta = (-2q^3+9q^2-8q)k^2+(6q^3+7q^2-26q+8)k+4q^4+18q^3+\frac{145}{4}q^2+27q-119 > 0. \tag{2}$$

# 3. Sketch of Theorem 3.2

Note that (2) is a quadratic inequality for $k$. Since $-2q^3 + 9q^2 - 8q < 0$, the inequality (2) has a solution only if

$$k < g(q),$$

where

$$g(q) = \frac{6q^3 + 7q^2 - 26q + 8 + f(q)}{4q^3 - 18q^2 + 16q},$$

and

$$f(q) = \sqrt{32q^7 + 36q^6 - 146q^5 - 776q^4 - 1032q^3 + 5936q^2 - 4224q + 64}.$$

It follows by $q \geq 10$ that $k < 3.3098\sqrt{q}$, i.e. the inequality (2) has a solution only if $k < 3.3098\sqrt{q}$. This contradicts $k \geq 3.3098\sqrt{q}$. So Theorem 3.2 is proved. □

## 4. Refine Theorem 3.2 for $4 \le q \le 9$

Let $U_1 = \{u_1, \dots, u_q\}$, $S = \{u_{q+1}, \dots, u_{4q-10}\}$ and $U_2 = V(P) \setminus (U_1 \cup S)$ (If $|P| < 4q - 10$, then let $U_1 = \{u_1, \dots, u_q\}$, $S = V(P) \setminus U_1$).

Define

$$\mathcal{I} = \{C \in \mathcal{F} \,|\, d^+(U_1, C) \ge q(q-1) + 1\},$$

$$\mathcal{O} = \{C \in \mathcal{F} \,|\, d^+(C, U_2) \ge 6q - 13\} \quad \text{and} \quad \mathcal{R} = \mathcal{F} \setminus (\mathcal{I} \cup \mathcal{O}).$$

• Let $C$ be a $q$-cycle. If there is a $q$-matching $M$ from $U_1$ to $C$, then $d^+(C, S) \le \frac{9}{4}q^2 - \frac{29}{4}q$.

Similarly, estimate the lower and upper bound of $d^+(\mathcal{F} \setminus \mathcal{O})$. We get all the possible cases: if $q = 4$, then $1 \le k \le 4$; if $9 \ge q \ge 5$, then $1 \le k \le 5$.

From the following statement, we finish the proof of the case $4 \le q \le 9$.

• Let $k$ be an integer with $k \le 5$. If there exist two cycles $C_1, C_2 \in \mathcal{F}$ such that $d^+(\{u_1, u_2\}, C_i) \ge 2q - 1$ for $i = 1, 2$, then we can extend $\mathcal{F}$.

## 4. Related Results and Open Problems –Cycle Factor in Digraphs

- For a subset $W \subseteq V(D)$, define

$$\delta^+(W) = \min \{d_D^+(v) : v \in W\},$$

$$\delta^-(W) = \min \{d_D^-(v) : v \in W\}.$$

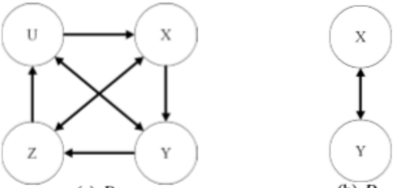
- **The minimum semi-degree of $W$ in $D$:**

$$\delta^0(W) = \min \{\delta^+(W), \delta^-(W)\}.$$

**Theorem 4.1 [Y. Wang and Y, 2019+]**

Suppose that $D$ is a digraph with order $n$ and $W \subseteq V(D)$. If $\delta^0(W) \geq (3n-3)/4$, then for any $k$ positive integers $n_1, \ldots, n_k$ with $n_i \geq 2$ for all $i$ and $\sum_{i=1}^k n_i \leq |W|$, $D$ contains $k$ disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap W| = n_i$ for each $i$.

- **A directed version of the Aigner-Brandt Theorem, when $W = V(D)$** [J. Lond. Math. Soc. 1 (1993) 39-51]:
  If $\delta(G) \geq (2n-1)/3$, then $G$ contains $k$ disjoint cycles of length $n_1, \ldots, n_k$, respectively. ($n \geq \sum_{i=1}^{k} n_i$ and $n_i \geq 3$ for all $i$)
- **Sharp** (in some sense)



(a) $D_1$ (b) $D_2$

$$D_1 : U = X = K^*_{4k-1},\ Y = Z = K^*_{4k}$$

$D_2 : X = K^*_{2k-1}$, $Y$ is an independent vertex set of order $k+1$.

**Conjecture 4.2 [Y. Wang and Y, 2019+]**

The minimum semi-degree in Theorem 4.1 can be improved to $2n/3$ when $n_i \geq 3$ for all $i$.

The degree condition is best possible by $D_2$.

It is supported by the following conjecture.

**Conjecture 4.3 [Czygrinow, Kierstead and Molla, Eur. J. Combin. 42 (2013) 1-14]**

If $n = 3k$ and $\delta^0(D) \geq 2k$, then $D$ contains $k$ disjoint $\triangle$s.

**Remark for Theorem 4.1.**

- When $k = 1$, $\delta^0(W) \geq (3n-3)/4 \Longrightarrow \delta^0(W) \geq \frac{n}{2}$
- Let $\lambda = \sum_{i=1}^{k} n_i$.
  If $n \geq 2\lambda$, then $\delta^0(W) \geq (3n-3)/4 \Longrightarrow \delta^0(W) \geq \frac{n}{2} + \lambda - 1$.


Thanks for your
attention !
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