Edge conditions on two disjoint cycles in graphs

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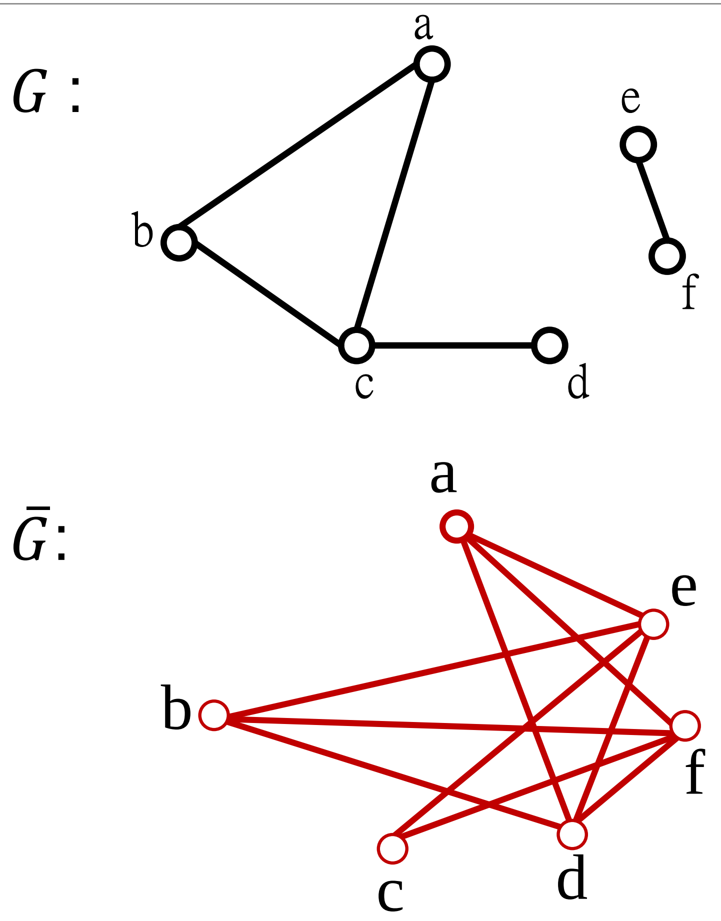
Outline

- Introduction
  - *Terminology and known theorems*
- Main results
  - *Theorem 3*
- Conclusion

Introduction-1

A **graph** $G, G = (V, E)$.

A **simple** graph is a graph with neither loops nor multiple edges.



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The **complement** of a given graph G *is denoted by* $\bar{G}$. $\bar{G} \equiv K_n - G$.

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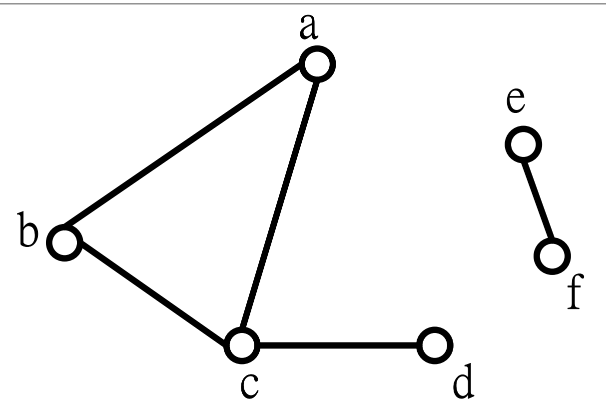
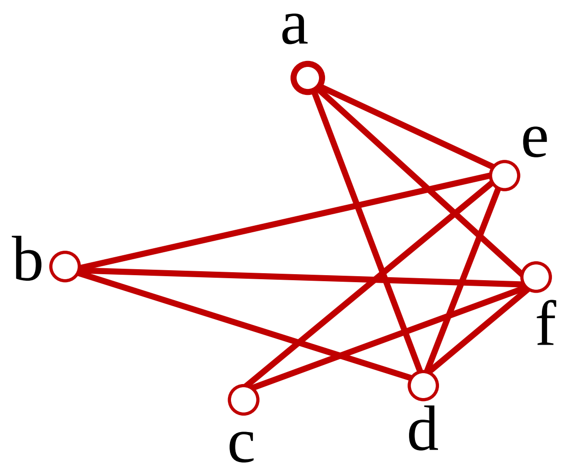


$\bar{G}$:



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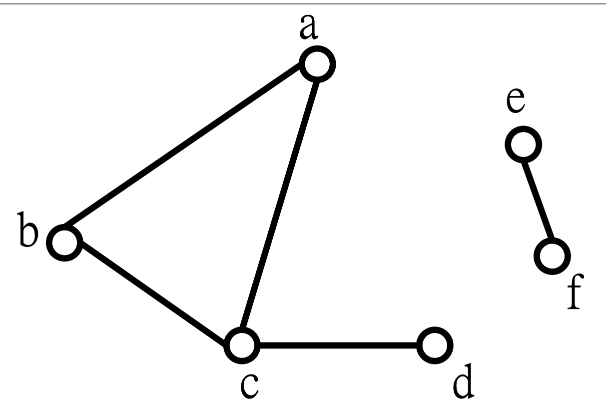
The **complement** of a given graph G *is denoted by* $\bar{G}$. $\bar{G} \equiv K_n - G$.

The **degree** of a given vertex u in G is defined by $\deg(u) = |\{v|(u, v) \epsilon E\}|$.

The **minimum degree** of the graph G is defined by $\delta = \min\{\deg(u)|u \epsilon V\}$.

G :



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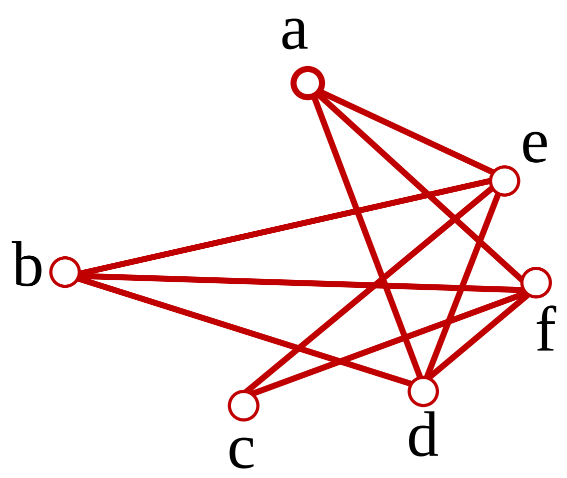
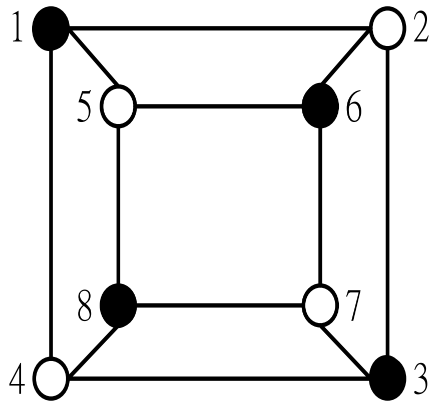


Introduction-2

Q: Under which conditions can we decompose a graph into two disjoint cycles with given lengths?



Minimum degree= δ =3;
4+4 decomposable;
Not 3+5 decomposable.

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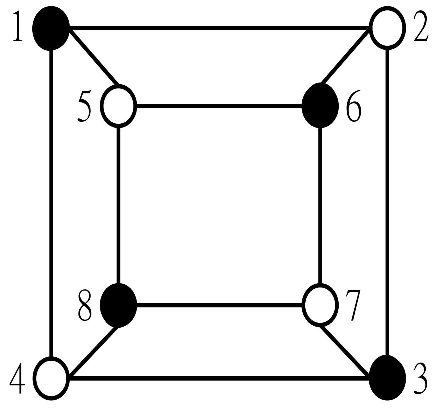

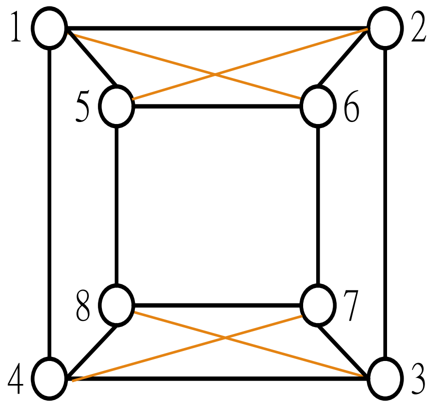

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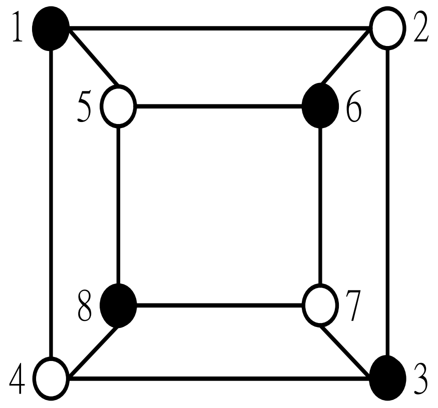

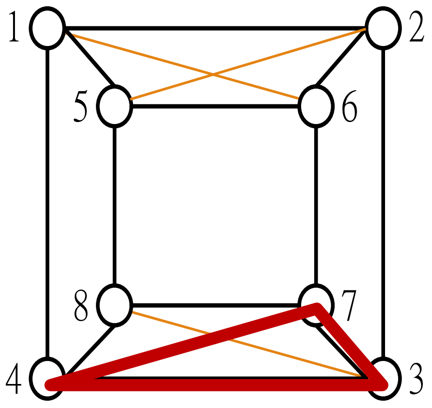

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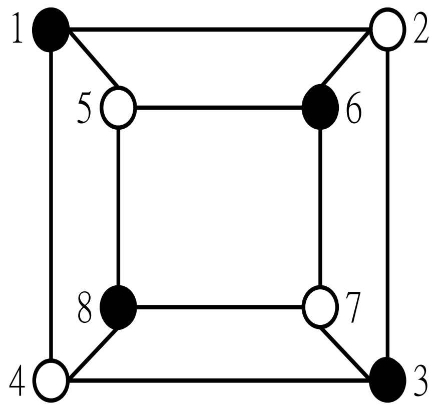

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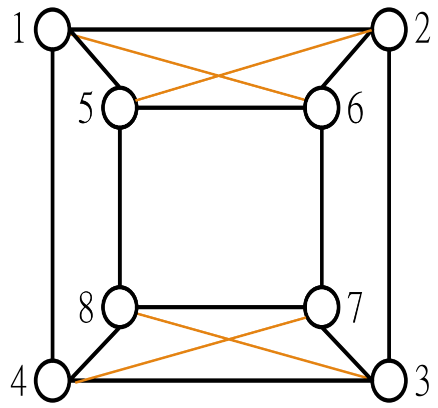

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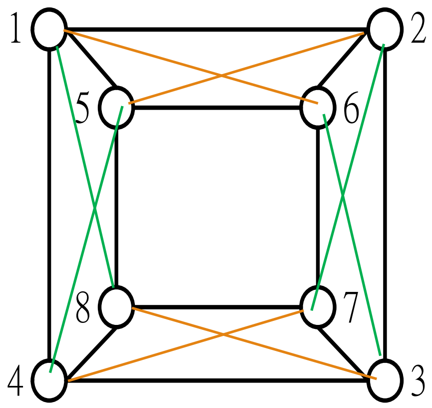

Minimum degree= δ =5;
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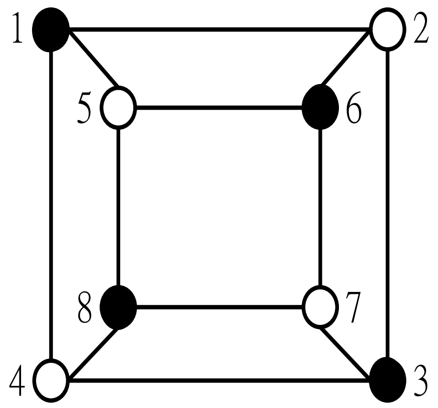

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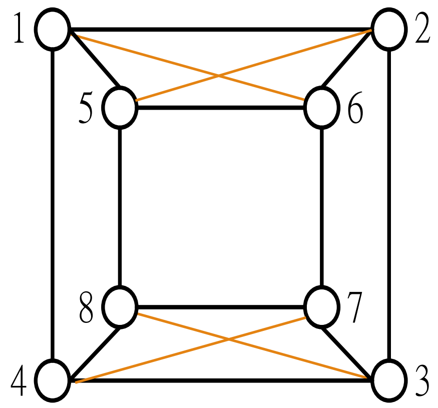

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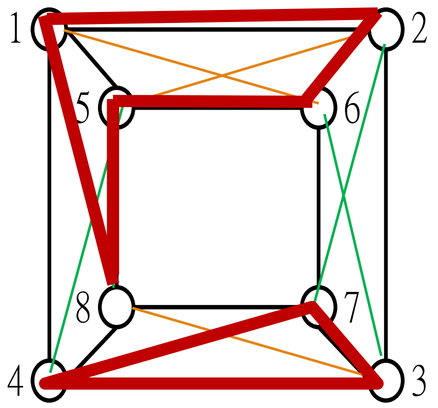

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Introduction-3

n1	n2	$\delta(G) \geq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil$
4	4	4
3	5	5

Theorem 1 *(M. H. El-Zahar, 1984, [2]) Let G be a graph with $|G| = n \geq 6$. Let n_1 and n_2 be two integers with $n_i \geq 3$ for $i = 1, 2$ and $n_1 + n_2 = n$. If $\delta(G) \geq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil$, then G has two disjoint cycles with lengths n_1 and n_2.*

Reference:

[2] M.H. El-Zahar, On circuits in graphs, Discrete Mathematics, Vol. 50, pp. 277–230, 1984.

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Theorem 2 *(J. Yan et. al., 2018, [5]) Let G be a graph with $|G| = n \geq 6$. Let n_1 and n_2 be two integers with $n_i \geq 3$ for $i = 1, 2$ and $n_1 + n_2 = n$. If $deg_G(u) + deg_G(v) \geq n + 4$ for every pair of nonadjacent vertices u and v of G, then G has two disjoint cycles with lengths n_1 and n_2.*

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[5] J. Yan, S. Zhang, Y. Ren and J. Cai, Degree sum conditions on two disjoint cycles in graphs, Information Processing Letters, Vol. 138, pp. 7–11, 2018.

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Introduction-4

Theorem 6. *Let G be a graph with $|G| = n \geq 6$. Let n_i be an integer with $n_i \geq 3$ for $i = 1, 2$, and $n_1 + n_2 = n$. If $\bar{e}(G) \leq n - 3$, then G contains two disjoint cycles with lengths n_1 and n_2.*

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Here $e = |E(G)|$, $\bar{e}$=$|E(\bar{G})|$. Obviously, $e + \bar{e} = \frac{n(n-1)}{2}$.

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Theorem 5. *(Hsu et.al., pp. 146, [3]) Let G be a graph with $n \geq 3$ and $\bar{e}(G) \leq n - 3$. Then G is hamiltonian. Moreover, the only non-hamiltonian graphs with $\bar{e}(G) \leq n - 2$ are $K_1 \circ K_1 \circ K_{n-2}$ and $\overline{K_2} \circ K_2 \circ K_1$.*

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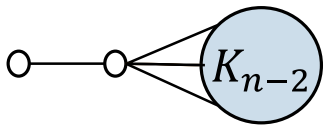

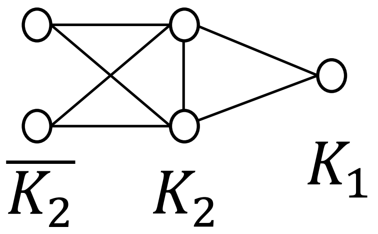

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Example. n=11. $\bar{e} \leq 8$.

n_1	n_2	$\bar{e}(G_1)$	$\bar{e}(G_2)$
3	8		
4	7		
5	6		

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4	7	$\leqq 1$	$\leqq 4$
5	6	$\leqq 2$	$\leqq 3$

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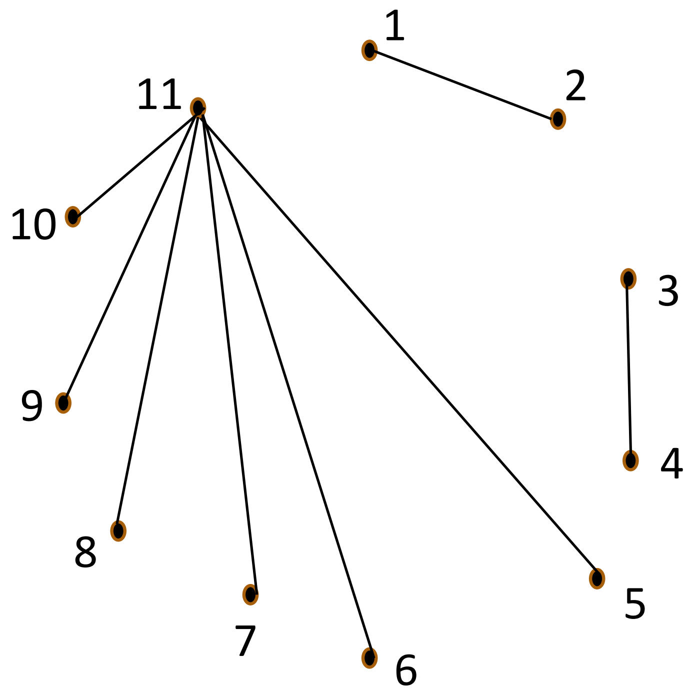
5	6	≦2	≦3



$\bar{G}$

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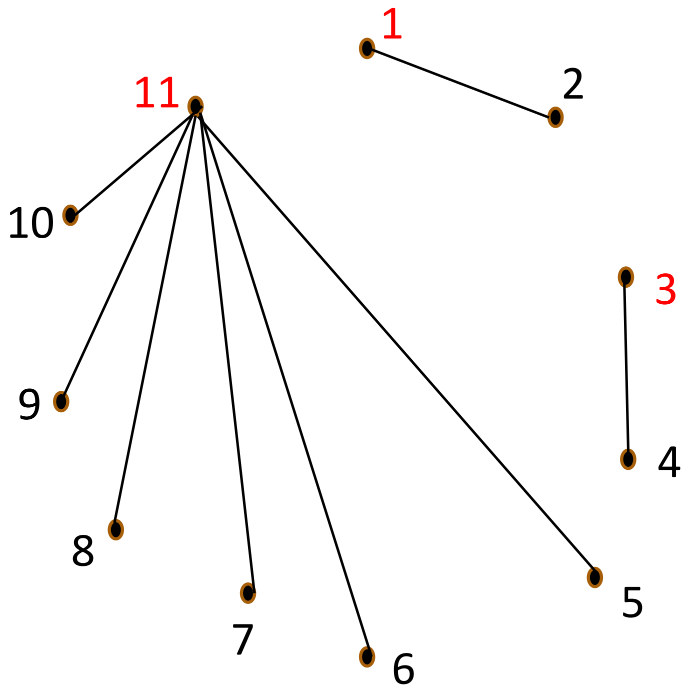

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Step 1. Given $n_1 = 3$, pick an independent set A_3 in $\bar{G}$ with 3 vertices such that the degree sum of these vertices is maximum. Let V(G_1)=A_3.

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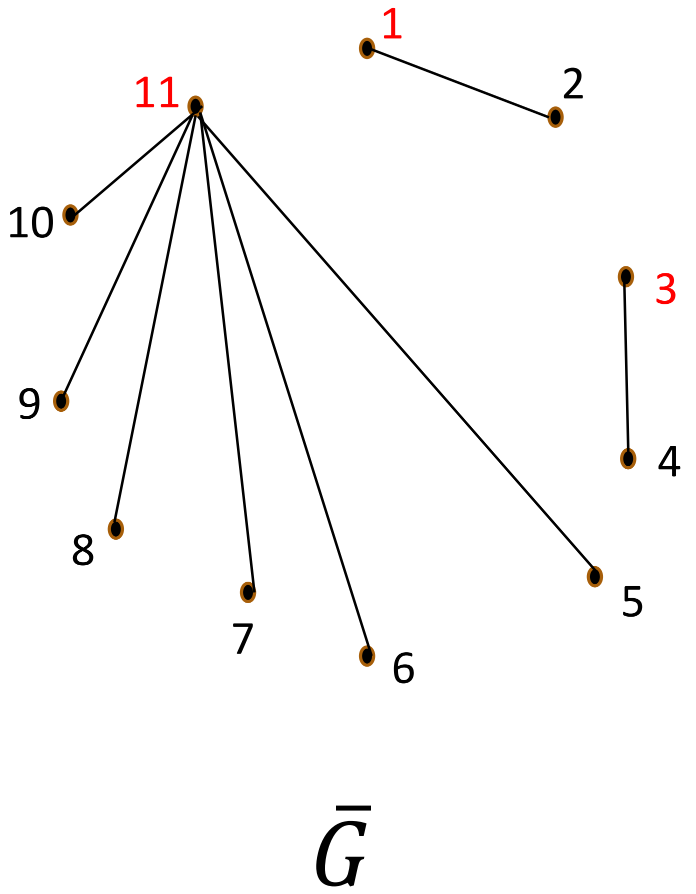

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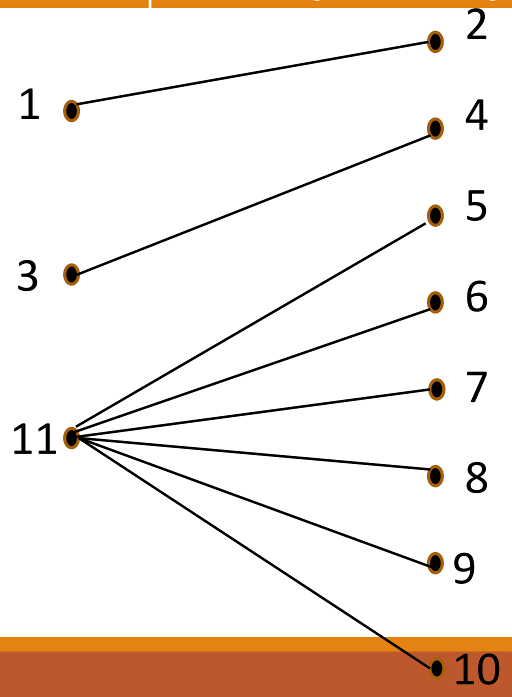
n_1=3	$A_3 = \{1, 3, 11\}$



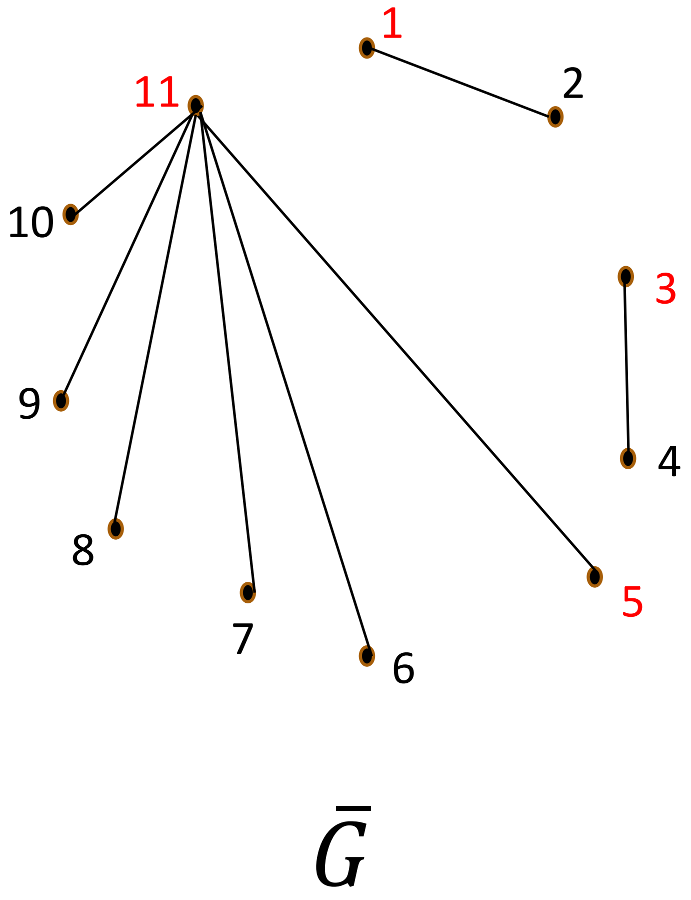
Thus $\bar{e}(G_1)$=0

$\bar{e}(G_2)$=0.

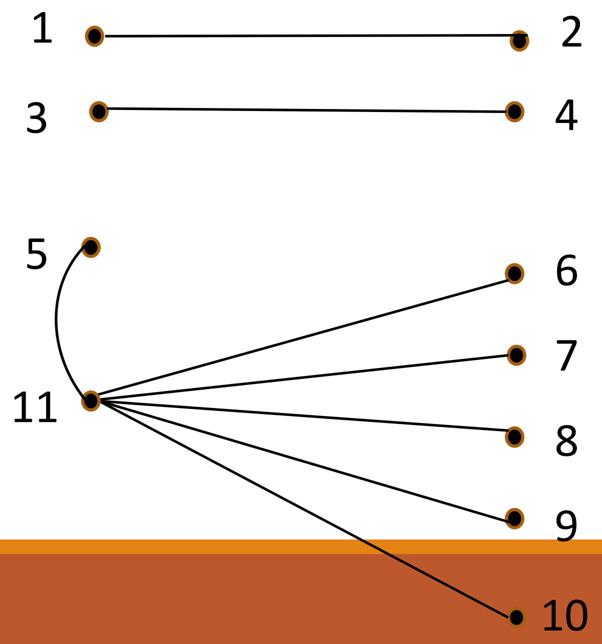
=> OK.

Example. n=11. $\bar{e} \le 8$.

Step 2. When $n_1 = 4$, pick an independent set A_4=$A_3 \cup \{x\}$ with $n_1 = 4$ vertices. If A_3 is already a maximal independent set in $\bar{G}$, then $A_4 = A_3 \cup \{x\}$ where x has the least neighbors in A_3. Let $V(G_1) = A_4$.



n_1=4	$A_4 = \{1, 3, 5, 11\}$

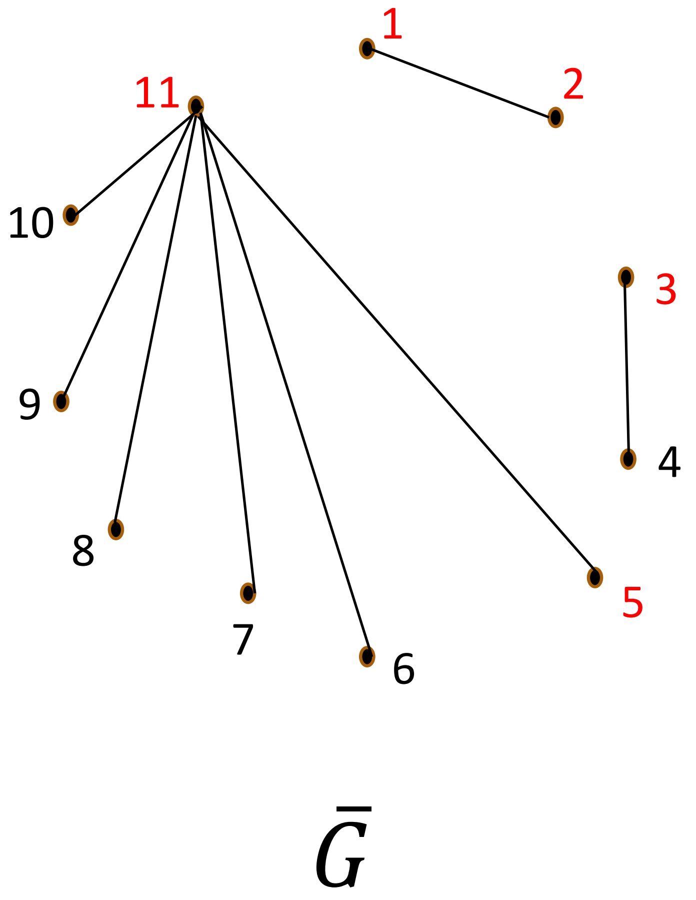


Thus $\bar{e}(G_1)$=1.

$\bar{e}(G_2)$=0.

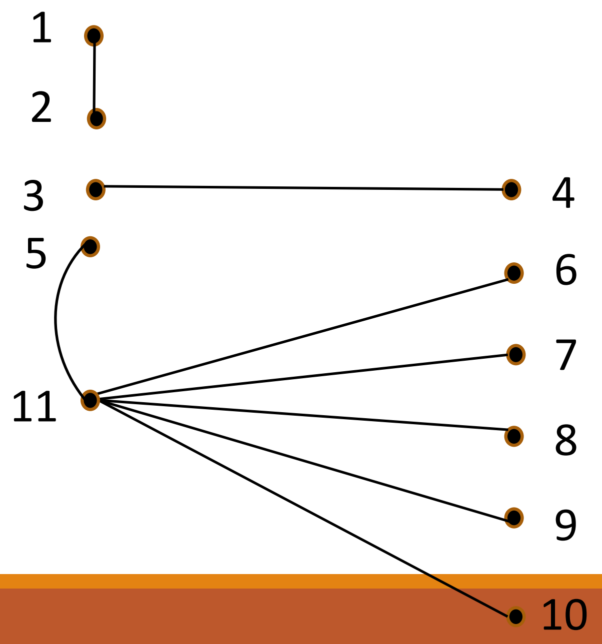
=> OK.

Example. n=11. $\bar{e} \leq 8$.



Step 2. When $n_1 = 5$, if A_4 is already a maximal independent set in $\bar{G}$, then $A_5 = A_4 \cup \{x\}$ where x has the least neighbors in A_4. Let V(G_1)=A_5.

n_1=5	$A_5 = \{1, 2, 3, 5, 11\}$



Thus $\bar{e}(G_1)$=2

$\bar{e}(G_2)$=0.

=> OK.

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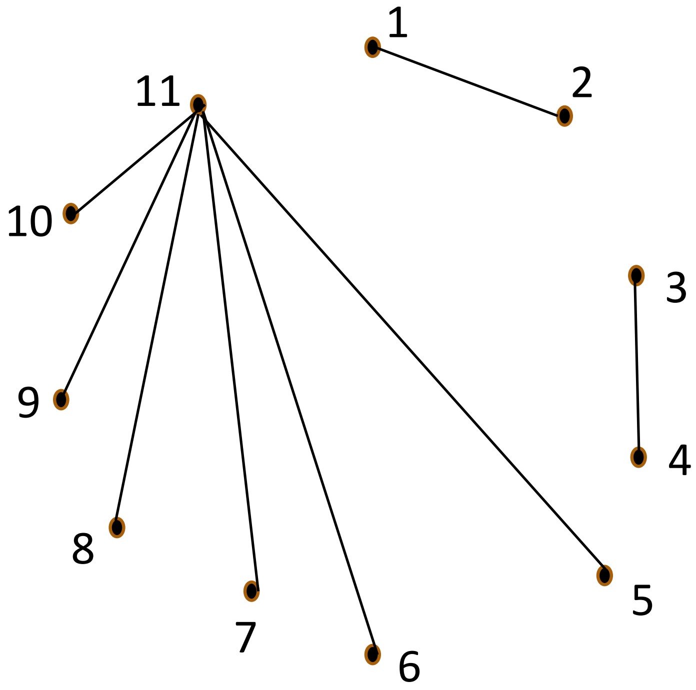
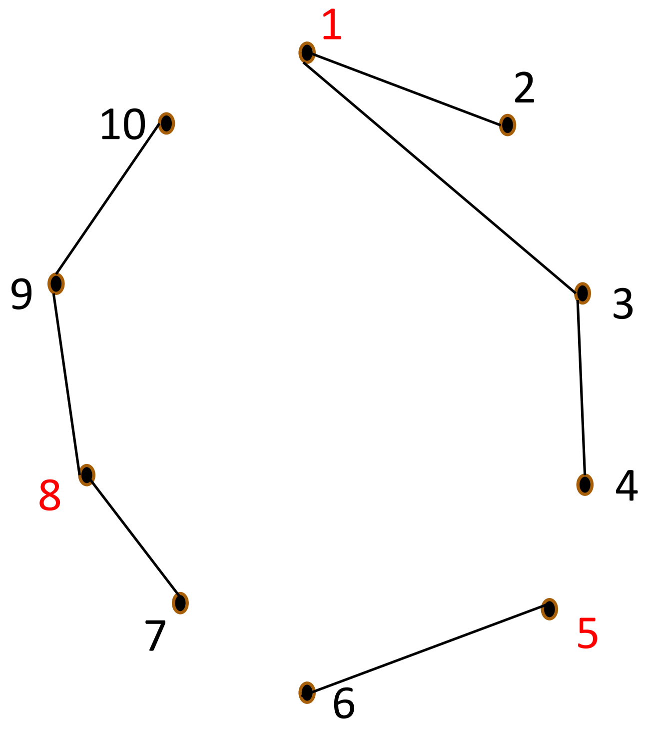
Example. n=11. $\bar{e} \leq 8$.



$\bar{G}$

n_1	n_2	$\bar{e}(G_1)$	$\bar{e}(G_2)$
3	8	0	$\leqq 5$
4	7	$\leqq 1$	$\leqq 4$
5	6	$\leqq 2$	$\leqq 3$

Ex1. n=10, $\bar{e} \leq 7$.



Complement(G)

Step 1. Given $n_1 = 3$, pick an independent set A_3 in $\bar{G}$ with 3 vertices such that the degree sum of these vertices is maximum. Let V(G_1)=A_3.

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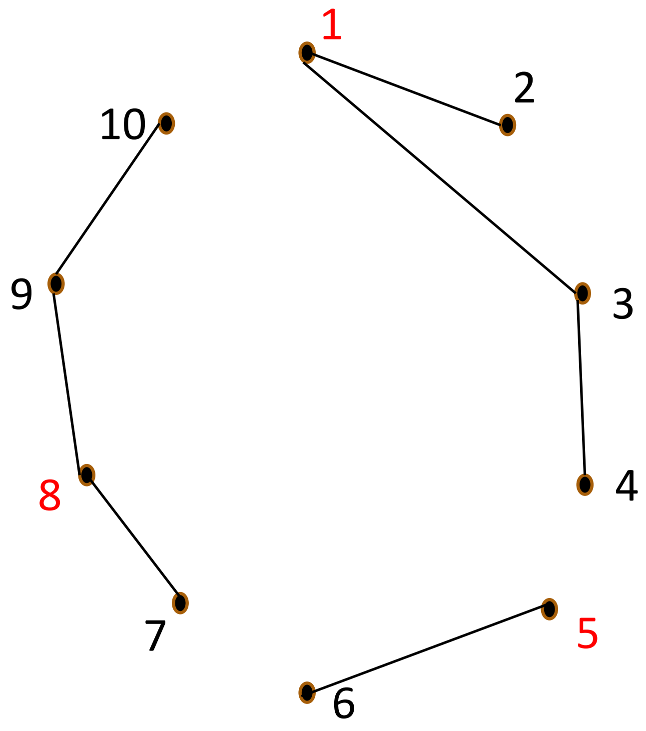

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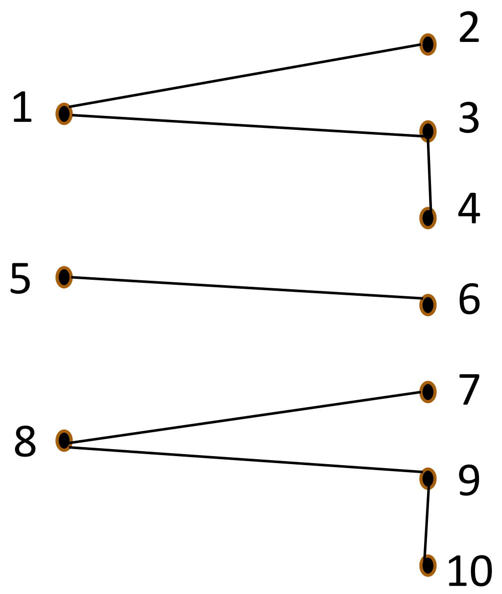
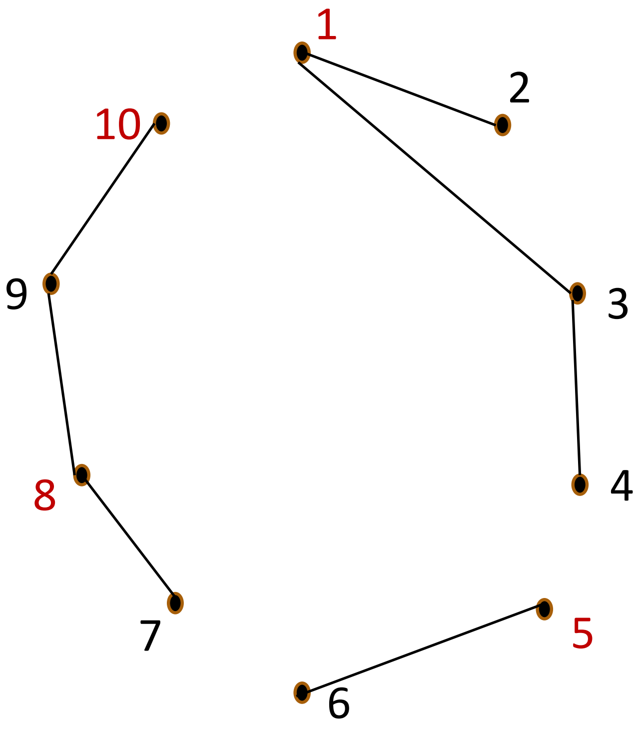


Thus $\bar{e}(G_1)$ =0

$\bar{e}(G_2)$=2 <=7-3

=>OK.

Ex1. n=10, $\bar{e} \leq 7$.



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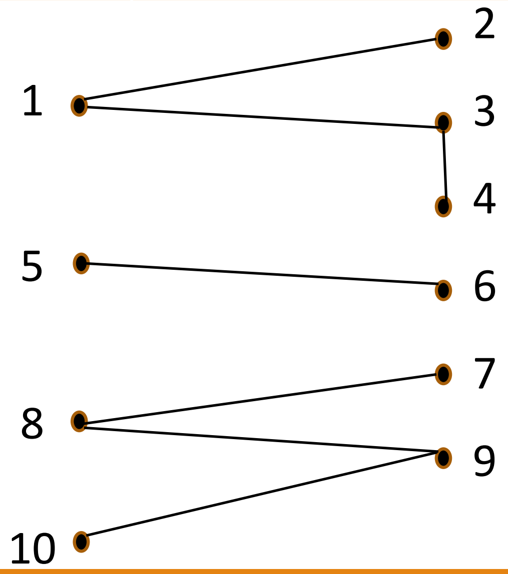
n_1=4	$A_4 = \{1, 5, 8, 10\}$

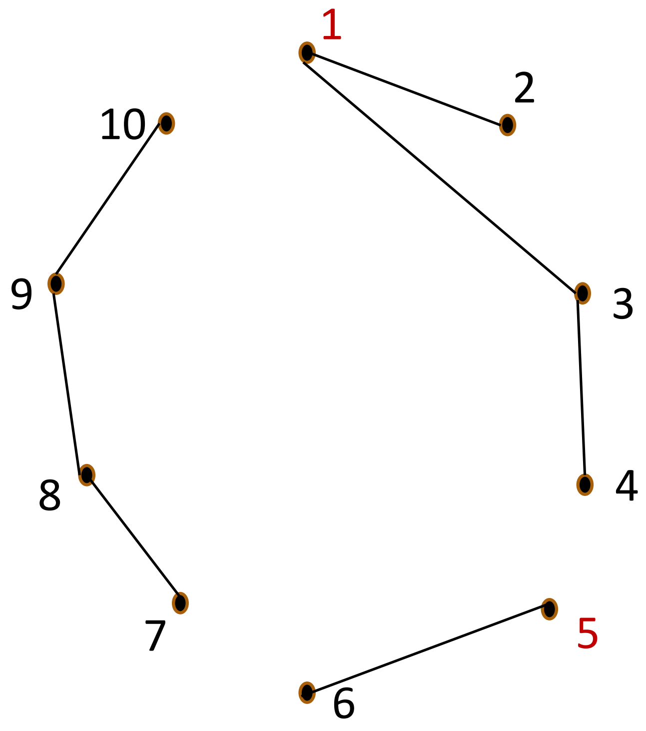


Thus $\bar{e}(G_1)$ =0

$\bar{e}(G_2)$=1 <=6-3.

=> OK.

Ex1. n=10. ebar(G)<=6



Complement(G)

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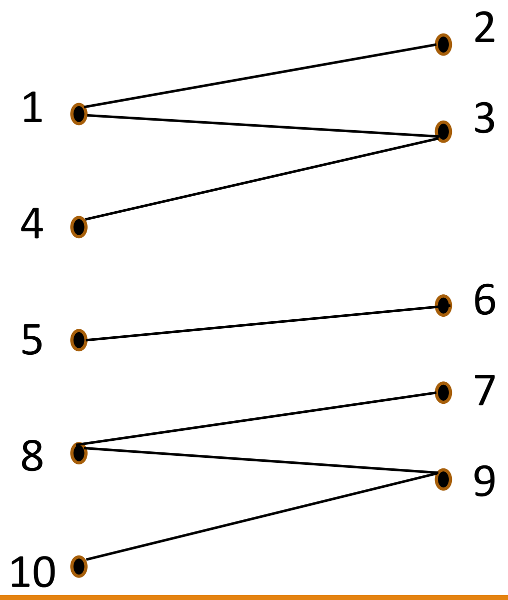
n_1=5	$A_5 = \{1, 2, 3, 5, 11\}$



Thus $\bar{e}(G_1)$=0

$\bar{e}(G_2)$=0.

=>OK.

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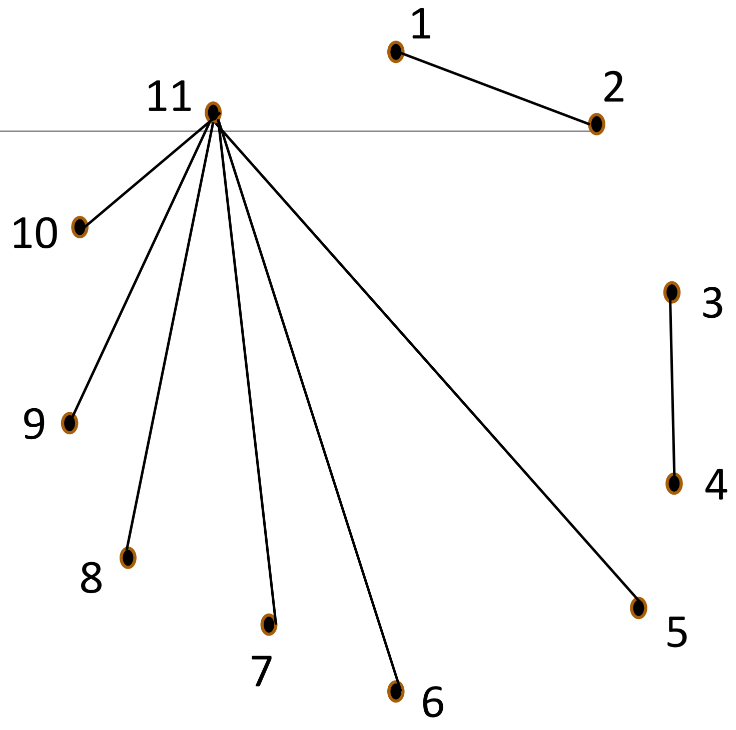
Ex. n=11. $e(\overline{G}) \leqq 8$

Conclusion

n_1	n_2	THM1. $\delta(G) \geq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil$	THM2. $deg_G(u) + deg_G(v) \geq n + 4$
3	8	6	15
4	7	6	
5	6	6	

Theorem 1 and Theorem 2 fail in this case,
but Theorem 6 is OK!



$\bar{G}$

δ=4

Minimum degree sum=13

Theorem 6. *Let G be a graph with $|G| = n \geq 6$. Let n_i be an integer with $n_i \geq 3$ for $i = 1, 2$, and $n_1 + n_2 = n$. If $\bar{e}(G) \leq n - 3$, then G contains two disjoint cycles with lengths n_1 and n_2.*

Cor. Let G be a graph with $|G| = n \geq 6$. Let $k \geq 2$ be an integer. Let $1 \leq i \leq k$ be an integer, n_i be an integer with $n_i \geq 3$, and $\sum_{i=1}^{k} n_i = n$. If $\bar{e}(G) \leq n - 3$, then G contains k disjoint cycles with length n_i.

Conclusion

With our theorem, we not only guarantee the existence of the two cycle decomposition of G, but also give the structure of the two subgraphs G_1 and G_2 and the required cycles.

As a result, our theorem outperforms THM 1 and THM 2 if edges of $\bar{G}$ are connected to a small number of vertices, where δ drops quickly and G has a wide spectrum of vertex degrees. However, THM 1 and THM 2 could perform better when the edges in $\bar{G}$ are more scattered such that the degree distribution of G is in better tune. Such a conclusion is true for general graphs with the total number of vertices $n \geq 6$.

~ the End ~

Thank you very much!