The crossing number of $K_{5,n+1} \setminus e$

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2019年8月

Definitions and Backgrounds

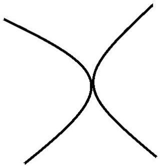
- A drawing of a graph G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs.

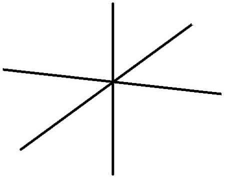
- A drawing is normal if it is satisfied the following conditions:

  (1) if two edges cross, they cross finite times;

  (2) there are no touching intersections;

  (3) no three edges cross at the same point.





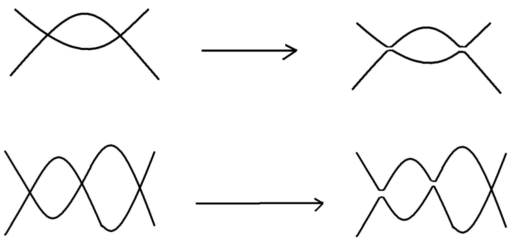
- Let ϕ be a normal drawing of a graph G. Denote by $cr_\phi(G)$ the number of crossings between edges of G under ϕ.
- The crossing number, $cr(G)$, of a graph G is defined a value as follows:

$$cr(G) = \min\{cr_\phi(G)\},$$

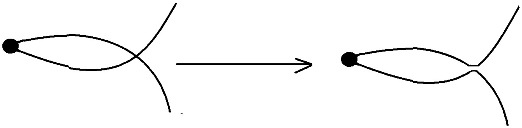
where the minimum is taken over all normal drawings ϕ of G.

A normal drawing with minimum number of crossings must satisfies the following conditions:

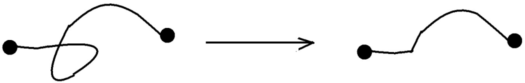
(1) if two edges cross, they cross at most once;



(2)adjacent edges do not cross;



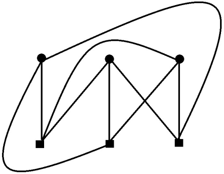
(3) edges do not have self-crossings.



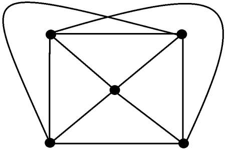
A normal drawing is good, if it is satisfied (1),(2) and (3) above.

By the definition of $cr(G)$:

- a graph G is planar $\iff cr(G) = 0$.
- $cr(K_{3,3}) = 1$, $cr(K_5) = 1$.



$K_{3,3}$



K_5

- the crossing number is an important parameter to measure how far a graph is from a planar graph.

The aspects in the study of the crossing number of graphs

- The exact determination of crossing number of some specific classes of graphs.
- Estimation of upper and lower bounds of the crossing number.
- The crossing number and the structural properties of graphs.
- The crossing number and other graph parameters.
- Various other forms of crossing numbers.
- The algorithm.
  (The crossing number problem is NP-complete !)
- The surface crossing number of graphs.

Two challenging conjectures

（1）The complete graph K_n

Conjecture (Guy, 1970's)

$$cr(K_n) = \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor :\triangleq Z(n)$$

For any real number x, $\lfloor x \rfloor$ means the largest integer not exceeding x.

- $0.8594Z(n) \leq cr(K_n) \leq Z(n)$. [Etienne, et.al., 2007]
- $cr(K_n) = Z(n)$ for $n \leq 10$. [Guy, 1972]
- $cr(K_n) = Z(n)$ for $n = 11, 12$. [Pan, Richter, 2007, JGT]
- (1) $cr(K_n) \geq \left\lceil \frac{n}{n-4} cr(K_{n-1}) \right\rceil$;

  (2) $cr(K_{13}) \in \{217, 219, 221, 223, 225\}$;

  (3) $cr(K_{13}) \neq 217$。

  [Dan McQuillan, Shengjun Pan, R.B.Richter, 2015, JCTB]

$cr(K_{13}) =?$

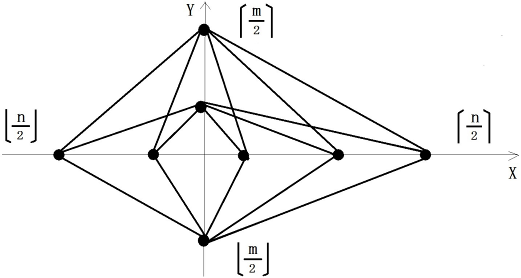
(2) The complete bipartite graph $K_{m,n}$ $(m \leq n)$

Conjecture (Zarankiewicz,1950's)

$$cr(K_{m,n}) = Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

There exists a good drawing achieving the conjectured value of the crossing number!

A drawing for $K_{3,5}$



- For $m \leq 6$，$cr(K_{m,n}) = Z(m,n)$. [Kleitman, 1970, JGT]
- Zarankiewcz's conjecture is true for $K_{7,7}$、$K_{7,8}$、$K_{7,9}$，$K_{7,10}$，$K_{8,8}$,$K_{8,9}$，$K_{8,10}$. [Woodall, 1993, JGT]

$cr(K_{7,n}) =$? for $n \geq 11$

Our motivation

The lower bound of $cr(G\backslash e)$ in terms of $cr(G)$.

- Every graph G contains an edge e so that $cr(G\backslash e) \geq \frac{2}{5}cr(G) - 1$. [Richter,Thomassen,1993, JCTB]
- Let G be a graph with no vertices of degree 3. Then there is an edge e of G such that $cr(G\backslash e) \geq \frac{1}{2}cr(G) - \frac{37}{2}$. [Salazar, 2000, JCTB]
- For every connected graph G with n vertices and $m \geq 1$ edges, and for every edge e of G, we have $cr(G\backslash e) \geq cr(G) - 2m + \frac{n}{2} + 1$. [J.Fox, C.D.Toth, 2015, JGT]

  (the above result improves on the Richter-Thomassen result for graphs with n vertices and $m \geq 10.1n$ edges.)

How to determine the exact value of $cr(G\backslash e)$ for a graph G whose $cr(G)$ is known ?

For example, $cr(K_{m,n}) = Z(m,n)$ for $m \leq 6$, what about $cr(K_{m,n}\backslash e)$ for any edge e of $K_{m,n}$.

[Gek L. Chia, and Chan L.Lee, Crossing number of nearly complete graph and nearly complete bipartite graph, ARS Combinatoria, 121 (2015),437-446.]

(1) $cr(K_{3,n}\backslash e) = Z^*(3,n)$, $cr(K_{4,n}\backslash e) = Z^*(4,n)$, where $Z^*(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$.

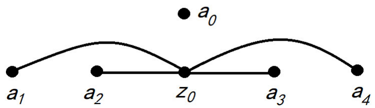
(2) $cr(K_{5,5}\backslash e) = Z^*(5,5)$.

Conjecture:
Let e be an edge in $K_{m,n}$. Then $cr(K_{m,n}\backslash e) = Z^*(m,n)$.

Our result

Theorem. $cr(K_{5,n+1} \backslash e) = n(n-1) = Z^*(5, n+1)$ for any $n \geq 0$.

Define H to be the following graph:

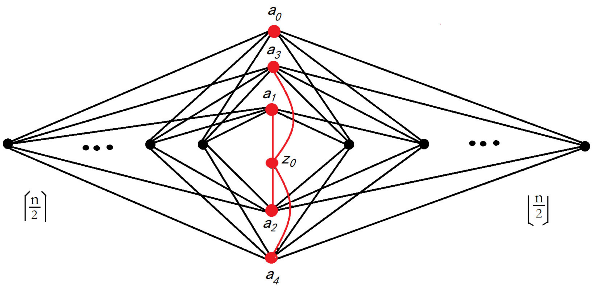


Define G_n to be the following graph: add $n (n \geq 1)$ new vertices $z_1, z_2, \cdots, z_n$, and connect each z_i $(1 \leq i \leq n)$ to all vertices of H except from z_0.

Obviously, $G_n \cong K_{5,n+1} \backslash e$

The sketch of the proof of Theorem

Lemma 1 (Upper bound). $cr(G_n) \le n(n-1)$.



$$cr_\phi(G_n) = Z(5,n) + 2\left\lfloor \frac{n}{2} \right\rfloor = n(n-1)$$

Lemma 2. $cr(G_n) = n(n-1)$ for $1 \leq n \leq 4$.

Because $G_1 \cong K_{5,2} \backslash e$, $G_2 \cong K_{5,3} \backslash e$, $G_3 \cong K_{5,4} \backslash e$, and $G_4 \cong K_{5,5} \backslash e$.

Our method is by induction on n.

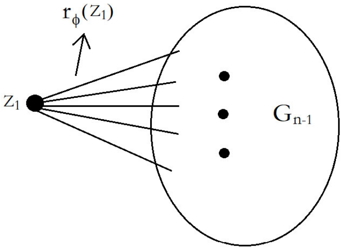
Suppose that $cr(G_k) = k(k-1)$ for any $1 \leq k \leq n-1$.

In order to prove that $cr(G_n) = n(n-1)$, by Lemma 1 it suffices to prove that $cr_\theta(G_n) \geq n(n-1)$ for any a good drawing θ.

Assume to contrary that G_n has a good drawing ϕ such that

$$cr_\phi(G_n) < n(n-1). \qquad (*)$$

Claim 1. For any $1 \le i \le n$, $r_\phi(z_i) \le 2n-3$, where $r_\phi(z_i)$ is the number of the crossings of ϕ involving the edges G_n incident to z_i.

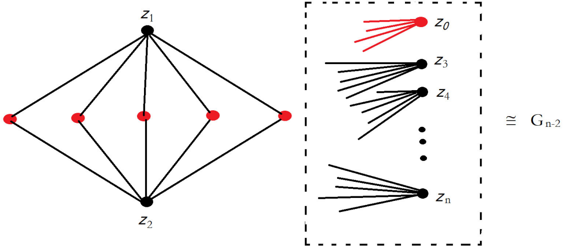


for,otherwise,

$$
\begin{aligned}
cr_\phi(G_n) &= r_\phi(z_1) + cr_\phi(G_{n-1}) \\
&\ge 2n-2+(n-1)(n-2) \\
&\ge n(n-1).
\end{aligned}
$$

A contradiction to $(*)$.

Claim 2. For any $1 \le i, j \le n$ and $i \ne j$, $cr_{\phi}(E_{z_i}, E_{z_j}) \ge 1$.



for,otherwise, $cr_{\phi}(G_n) =$

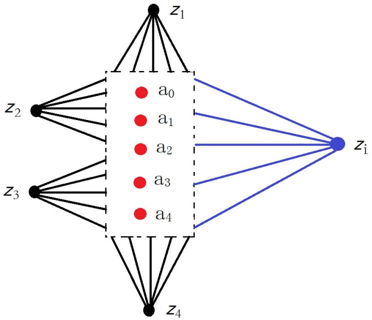
$$cr_{\phi}(E_{z_1} \cup E_{z_2}, E_{z_0}) + cr_{\phi}(E_{z_1} \cup E_{z_2}, \bigcup_{i=3}^{n} E_{z_i}) + cr_{\phi}(G_{n-2}) \ge n(n-1).$$

A contradiction to $(*)$.

Claim 3. There exist four distinct Z-vertices of G_n, say z_1,z_2, z_3, and z_4, such that

(1) $cr_\phi(E_{z_1} \cup E_{z_2} \cup E_{z_3} \cup E_{z_4}) = 9$.

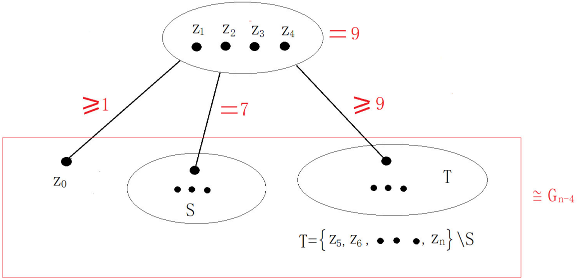
(2) for any $5 \leq i \leq n$, $cr_\phi(E_{z_1} \cup E_{z_2} \cup E_{z_3} \cup E_{z_4}, E_{z_i}) = 7$, or ≥ 9; moreover, if $cr_\phi(E_{z_1} \cup E_{z_2} \cup E_{z_3} \cup E_{z_4}, E_{z_i}) = 7$, then $cr_\phi(E_{z_1}, E_{z_i}) = 4$.



For otherwise, we can also induce a contradiction to $(*)$.

Now estimate $cr_\phi(G_n)$.

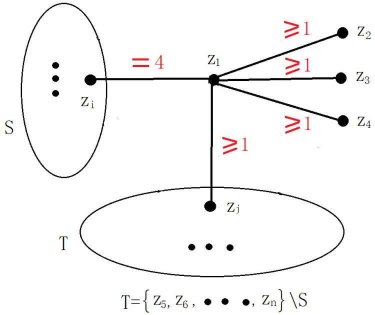
Set $S = \{z_i \mid cr_\phi\left(\bigcup_{k=1}^{4} E_{z_k}, E_{z_i}\right) = 7, 5 \le i \le n\}$ and $|S| = s$.



So, by the above Claims and the inductive hypothesis,

$$\begin{aligned} cr_\phi(G_n) \ge& 9 + 1 + 7s + 9(n-4-s) + cr_\phi(G_{n-4}) \\ =& 9n - 2s - 26 + (n-4)(n-5) = n^2 - 2s - 6 \end{aligned}$$

Now estimate $r_\phi(z_1)$.



By the definition of $r_\phi(z_1)$, and the above Claims,

$$r_\phi(z_1) \geq 3 + 4s + (n - 4 - s)$$
$$= n - 1 + 3s$$

By Claim 1, $r_\phi(z_1) \leq 2n-3$, and thus $n-1+3s \leq 2n-3$. So,

$$3s \leq n-6.$$

By the above arguments,

$$cr_\phi(G_n) \geq n^2 - 2s - 6.$$

Note that $n \geq 5$, and we can obtain that $cr_\phi(G_n) \geq n(n-1)$, a contradiction to $(*)$.

Therefore, $cr(G_n) = n(n-1)$, proving the theorem.

Remark: Recently we have determined $cr(K_{3,n} \backslash 2e)$, $cr(K_{4,n} \backslash 2e)$, $cr(K_{5,n} \backslash 2e)$.

But $cr(K_{6,n} \backslash e) = ?$, $cr(K_{6,n} \backslash 2e) = ?$

THANK YOU !